# NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XVIII — Baton Rouge, La., November, 1943 — No. 2



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VOL. XVIII BATON ROUGE, LA., November, 1943 No. 2

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.



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# A Plea for Analytical Mechanics

The National Defense program naturally and properly suggests that mathematicians devote more attention at this time to applied mathematics. While the writer of this paper readily falls in line with this suggestion yet, in making his plea for greater emphasis on analytical mechanics, candor compels him to say that he is doing no more than expressing convictions formed many years ago and altogether regardless of war. The stimulus of present agitation and the unusual activity with reference to national defense are welcomed as providing powerful incentive and opportunity to press the claims of analytical mechanics. Unquestionaly it is a desirable study, not alone because of its untold fundamental value in applied science, but far more as worthy of a place in any well-rounded mathematical curriculum, being intensely interesting, inspirational, educational, cultural. Even though there were no science of gunnery and no need of a knowledge of ballistics for war preparedness, it would still remain true that the subject of analytical mechanics should be cultivated for its own sake. It has long since won a prominent place in its own right. By every token, that position should be protected and preserved. The special plea of this paper is not made from the utilitarian standpoint at all but rather by reason of intrinsic merit.

Analytical mechanics is a fascinating study. Many of its facts are striking, its principles and propositions beautiful. How impressive and remarkable the generalizations of the great masters in both pure and applied mathematics!

A knowledge of the elements of mechanics is the logical foundation of physical science. It goes without saying, then, that, in this broad and ever-widening field, a thorough understanding of the principles of mechanics is essential. Fortunately there is no dearth of excellent texts. Books are available ranging from the elementary which are level to the comprehension of college freshmen to the more advanced works intended for juniors and seniors majoring in mathematics in the

colleges of arts and science as well as those classic treatises which will test and tax the capacity of the strongest students of the graduate schools.

Students in England and in some of the European countries have long been given thorough courses in elementary statics and dynamics before taking up the studies of analytical geometry and calculus. Then, with these subjects well in hand, advanced courses in analytical statics, kinetics of a particle, and rigid dynamics follow. All this is in somewhat distressing contrast with American procedure.

Time was, though quite a good while ago, when some of the finest courses of advanced work in mathematics as afforded by American colleges and universities were in mechanics, astronomy, and other physical sciences. These were offered without any sort of apology right alongside of courses in pure mathematics. Experience of those days indicated conclusively that these offerings were among the very best, to say the least of it.

It appears evident that interest in analytical mechanics on the part of American mathematicians has been on the wane for a decade or more, perhaps for a quarter of a century. This field of study has been neglected in favor of the abstruse and abstract. Evidence of the truth of these statements is not wanting. It may be gathered from reviews of recent publications, current periodical literature, lists of thesis subjects for M. A. and Ph.D. degrees, and programs of mathematics clubs. These sources of information furnish little or nothing relating to theoretical mechanics. And the same remark applies with almost equal force to college catalogues listing courses, either required or elective, for the B. A. degree. Furthermore, it is not difficult to recall the day when *some first-class mathematical periodicals carried fine sections of problems in mechanics*. Unhappily, that feature has largely, if not wholly, disappeared. Why not hasten a return and restoration of the prominence formerly enjoyed by analytical mechanics?

Certainly there is a revival of interest in the science of mechanics as applied to engineering. This is easily seen in the splendid technical schools, engineering colleges, and institutes of technology. More time and greater emphasis are being given to theoretical as well as applied mechanics in these institutions. But engineering education, while dependent on this important subject, has no desire to monopo-

lize it. The colleges of arts and science should offer it to such students as can take it.

It is admitted that the student of analytical mechanics is bound to meet with very difficult and complex problems and principles which must be mastered. Strong courses in analytics and calculus will fit him for the strenuous efforts which lie ahead.

It seems much to an aspiring student to come in contact with the powerful minds and massive intellects of an older day. It means much to lay hold, even though feebly, of the immortals, to call the names of Lagrange with his marvelous generalizations (witness his *Mecanique Analytique*), of Laplace with his wonderfully profound theorems of physical astronomy (see his *Celestial Mechanics*), of Newton, Euler, d'Alembert with their sublime works.

It may be that most men, including the writer, do little more than catch glimpses of these giants of the past. We may not be able to fully appreciate their keenly analytical minds, yet the realm in which they held sway and still hold supremacy commands respect and elicits unbounded admiration.

In view of the tremendous expansion of modern mathematics with new and constantly growing fields for research, one wonders if the time is not ripe for the investigator to re-enter the attractive region of analytical mechanics. Surely the last word on that subject has not been said. May it not be that more and even brighter light is yet to break and still grander truth is yet to be revealed? The volume of this great book is not yet completed and closed. Some soaring genius may some day enter this absorbing field if only mathematicians of today will offer incentive, encouragement, and opportunity.

*University of Mississippi.* ALFRED HUME.

# On Derived Sets*

By N. E. Rutt
*Louisiana State University*

It is the intention of this paper to adopt an attitude toward the concept of the derived set farther removed than customary from the notion in its elementary form, and to investigate some of those properties of this weakened concept which project themselves into the commoner characteristics of the usual derived set. There is some tendency among mathematicians of rigid views to classify subdivisions of their science upon a basis of respectability, and to hide those of certain grades behind closed doors where they will be less likely to disturb the sensibilities of chance students straying in from other fields. No presumption of agreement or of disagreement with such scholars is intended here, but it seems only fair to cooperate with them. To this end a warning is now posted. It can not be asserted that what is about to follow deserves general respect. But it is hoped that, to those who are diverted by logical processes either with or without flaws, the following samples may be of interest.

The basis of the discussion is a space composed of the points of the class $S$, the collection $[S]$ of all the sub-classes or subsets of $S$, and a law £ which associates with each member of $[S]$ another one of $[S]$. The law has upon the subsets of $S$ an ordered effect of pairing each of them off with another one like the effect of a set-valued single-valued function of the members of $[S]$, the arguments of the function being the members of $[S]$ and the values of the function being members of $[S]$, the range of the function being the entire set $[S]$ but the inverse of the function being not necessarily single-valued.

1:1 *Definitions.* If the law £ associates with the element $S_0$ of $[S]$ the element $S_1$ if $[S]$, then $S_1$ is called the ***first derived set*** of $S_0$ or merely the ***derived set*** of $S_0$.† The ***second derived set*** $S_2$ of $S_0$ is the first derived set of the first derived set of $S_0$. In general, if $n$ is zero, the $n$th derived set of $S_0$ is $S_0$, while if $n$ is a natural number, ***the nth derived set*** $S_n$ of $S_0$ is the first derived set of $S_{n-1}$. The element $A$ of $[S]$ is ***closed of grade n***, $n$ a natural number, if $A$ contains every member of $[S]$ which is the $n$th derived set of some subset of $A$. The element $A$ of $[S]$ is ***closed*** if it is closed of grade 1.

*Reported in part to the Society during December, 1940.

†This version of the notion of derived sets, as well as the following definition of closed set, has been used for several years by H. L. Smith.

1:2 *Comments.* Owing to the arbitrary behavior of £, the notion of closed set as ordinarily defined proves too feeble in $S$ to imply anything of general interest. This is the reason for the definition above. Upon the basis of it there will be proved in what follows some statements of familiar sound such as: (1) the product of two sets each closed of grade $n$ is also closed of grade $n$, and (2) if $[M_i]$ is a monotone decreasing sequence of sets each closed of grade $n$, then the set of points common to all members of the sequence is closed of grade $n$.

2:1 Theorem. *A set closed of grade $n$ is also closed of grade $kn$, $k$ being any natural number.*

*Proof.* Let $Q$ be any subset of $A$, and let $R$ be the $n$th derived set $Q_n$ of $Q$. As $A \supset Q_n = R$ by supposition, then $R$ is a subset of $A$ whose $n$th derived set $R_n$ is accordingly in $A$. But $R_n$ is the $2n$th derived set of $Q$. As $A \supset Q_{2n}$ for each subset $Q$ of $A$, then $A$ is closed of grade $2n$. By supposing $Q$ to be any subset of $A$ and letting $R$ be the $(k-1)n$th derived set of $Q$, an induction may be completed along similar lines.

2:2 Corollary. *If the set $A$ is closed of grade $n$ and if its collection of subsets is $[A_0]$ and if $[B_0]$ is a subcollection of $[A_0]$, then the $kn$th derived set of each of $[B_0]$ is one of $[A_0]$.*

3:1 Theorem. *If the set $A$ is closed of grade $n$ and the set $B$ is also closed of grade $n$, then the set $A \cdot B$ is likewise closed of grade $n$.*

*Proof.* Let $Q$ be any subset of $A \cdot B$ and let $Q_n$ be the $n$th derived set of $Q$. As $A \supset Q$ and $A$ is closed of grade $n$ then $A \supset Q_n$. As $B \supset Q$ and $B$ is closed of grade $n$ then $B \supset Q_n$. Accordingly $A \cdot B \supset Q_n$. Since $A \cdot B$ contains the $n$th derived set of any set, such as $Q$, contained in $A \cdot B$, then $A \cdot B$ is closed of grade $n$.

4:1 Theorem. *If the set $A$ is closed of grade $k$ and the set $B$ is closed of grade $h$ and if $m$ is the least common multiple of $k$ and $h$, then the product $A \cdot B$ is closed of grade $m$.*

*Proof.* By 2:1 the set $A$ is closed of grade $m = k \cdot K$, $K$ a natural number. Similarly the set $B$ is closed of grade $m = h \cdot H$, $H$ a natural number. Accordingly $A \cdot B$ is closed of grade $m$ by 3:1.

5:1 Notation. *In what follows the collection of all subsets of the set $A$ will be represented by $[A_0]$, the collection of all sets which are $j$th derived sets of members of $[A_0]$ by $[A_j]$, and the collection of all sets which are $i$th derived sets of members of $[A_j]$ by $[A_{j+i}]$.*

5:2 Theorem. *If the set $A$ is both closed of grade $h$ and also closed of grade $k$ and if $d$ is the greatest common divisor of $h$ and $k$, there is a*

*natural number $r$ such that all members of $[A_r]$ and all members of $[A_{r+d}]$ belong to $[A_0]$.*

*Proof.* Since non-zero integers $a$ and $b$ exist, in fact $|a|<k$ and $|b|<h$, such that $ah+bk=d$, and since $a$ and $b$ have opposite signs, either (1) $ah-|b|k=d$ and accordingly $ah=|b|k+d$, or (2) $-|a|h+bk=d$ and accordingly $bk=|a|h+d$.

Now the members of $[A_{|b|k}]$ are, in case (1), members of $[A_0]$ because of 2:1, the set $A$ being closed of grade $k$. Also the $d$th derived sets of the elements of $[A_{|b|k}]$are the members of $[A_{|b|k+d}]$, or the members of $[A_{ah}]$ accordingly. But $[A_{ah}]$ is a subcollection of $[A_0]$ because of 2:1 and the fact that $A$ is closed of grade $h$. Thus in case (1) every one of $[A_{|b|k}]$ is in $[A_0]$ and has its $d$th derived set also in $[A_0]$. In case (1) therefore $r=|b|k$ satisfies the theorem.

In case (2) $r=|a|h$ satisfies the theorem by means of similar reasoning.

5:3 Corollary. *If the set $A$ is closed of grade $p$ and also closed of grade $q$ where $p$ and $q$ are relatively prime, then there is a natural number $r$ such that both of the collections $[A_r]$ and $[A_{r+1}]$ are subcollections of $[A_0]$.*

5:4 Corollary. *If the set $A$ is closed of grade $p$ and also closed of grade $q$ where $p$ and $q$ are relatively prime, then there are subsets of $A$ whose derived sets are in $A$.*

5:5 *Examples.* Example (1). The set $A$ consists of the two points $a$ and $b$, and the set $[A_0]$ consists of the four sets $(a,b)$, $(a)$, $(b)$, $(0)$. Arrows indicate the successive derived sets.

$$
\begin{array}{lllllllllll}
(a,b) & \to & u_1 & \to & u_2 & \to & u_3 & \to & (0) & \to & \cdots \\
(a) & \to & u_4 & \to & (b) & \to & u_5 & \to & (a) & \to & \cdots \\
(b) & \to & u_5 & \to & (a) & \to & u_4 & \to & (b) & \to & \cdots \\
(0) & \to & u_6 & \to & u_7 & \to & u_8 & \to & (a,b) & \to & \cdots
\end{array}
$$

Example (2). The set $A$ consists of the three points $a$, $b$, and $c$, and the set $[A_0]$ consists of the eight sets $(a,b,c)$, $(a,b)$, $(a,c)$, $(b,c)$, $(a)$, $(b)$, $(c)$, and $(0)$.

$$
\begin{array}{lllllllllllll}
(a,b,c) & \to & u_1 & \to & u_2 & \to & (b) & \to & u_9 & \to & (a) & \to & \cdots \\
(a,b) & \to & u_3 & \to & u_4 & \to & (b) & \to & u_9 & \to & (a) & \to & \cdots \\
(a,c) & \to & u_5 & \to & u_6 & \to & (c) & \to & u_{10} & \to & (a) & \to & \cdots \\
(b,c) & \to & u_7 & \to & u_8 & \to & (c) & \to & u_{10} & \to & (a) & \to & \cdots \\
(a) & \to & (a) & \to & (a) & \to & (a) & \to & (a) & \to & (a) & \to & \cdots
\end{array}
$$

$$(b) \;\to u_9 \to (a) \to (a) \to (a) \to (a) \to \cdots$$

$$(c) \;\to u_{10} \to (a) \to (a) \to (a) \to (a) \to \cdots$$

$$(0) \;\to (b) \to u_9 \to (a) \to (a) \to (a) \to \cdots$$

5:6 *Comments.* In the examples the symbols $u_1, \cdots, u_{10}$ represent subsets of $S$ not belonging to $[A_0]$. The set of example (1) is closed of grade 4. The set of example (2) is closed of grade 3 and also closed of grade 5. It is not closed of grade 1, but there is a natural number $n$, $n=5$, such that $[A_0] \supset [A_n] \supset [A_{n+1}]$.

6:1 Theorem. *If the set $A$ is closed of grade $h$, then $[A_0]$, $[A_h]$, $[A_{2h}], \cdots, [A_{(n-1)h}]$, $[A_{nh}], \cdots$ is a monotone decreasing sequence of collections of sets, that is, for each natural number $n$, the members of the collection $[A_{nh}]$ are included among the members of the collection $[A_{(n-1)h}]$.*

*Proof.* If the set $B$ is a member of $[A_h]$ then the $h$th derived set of $B$ is also in $[A_h]$, for, being in $[A_h]$, then $B$ is necessarily also in $[A_0]$, and every member of $[A_0]$ has an $h$th derived set in $[A_h]$. In consequence $[A_0] \supset [A_h] \supset [A_{2h}]$. Furthermore if

$$[A_0] \supset [A_h] \supset [A_{2h}] \supset \cdots \supset [A_{nh}],$$

and $B$ is one of $[A_{nh}]$ then the $h$th derived set of $B$ is also in $[A_{nh}]$, for by inductive hypothesis $B$ is in $[A_{(n-1)h}]$ and all $h$th derived sets of members of $[A_{(n-1)h}]$ are in $[A_{nh}]$. Accordingly $[A_{nh}] \supset [A_{(n+1)h}]$, and the induction is complete.

7:1 Theorem. *If the set $A$ is closed of grade $h$, then the collection, for all natural numbers, of its sets of derived sets is a collection of $h$ sequences, each of which is monotone decreasing.*

*Proof.* By 6:1 the sequence $[A_0]$, $[A_h]$, $[A_{2h}], \cdots$ is monotone decreasing. Because of this the sequence $[A_1]$, $[A_{h+1}]$, $[A_{2h+1}], \cdots$ is also monotone decreasing. By induction $[A_i]$, $[A_{h+i}]$, $[A_{2h+i}], \cdots$ is similarly monotone decreasing, $i$ being any natural number less than $h$.

8:1 Theorem. *If the set $A$ is closed of grade $h$ and also closed of grade $k$, and $d$ is the greatest common divisor of $h$ and $k$, there is a natural number $r$ such that the collection $[A_0]$ contains the collection $[A_{r+nd}]$ for all of the values* 0,1,2,3, $\cdots$ of $n$.

*Proof.* As in the proof of 5:2 there are non-zero integers $a$ and $b$ such that either (1) $ah=|b|k+d$ or (2) $bk=|a|h+d$.

In case (1) let $s=|b|k$. Then $[A_s] \subset [A_0]$ and $[A_{s+d}] \subset [A_0]$ because of 5:2. Let $[A_{s+d}]$ be the subset $[B_0]$ of $[A_0]$. Then $[B_s]$

is a subset of $[A_s]=[A_{|b|k}]$, so $[B_s]$ is a subset of $[A_0]$ because $[A_0]$ contains $[A_s]$, and furthermore $[B_s]$ is $[A_{2s+d}]$. Moreover $[B_{s+d}]$ is a subset of $[A_{s+d}]=[A_{|b|k+d}]=[A_{ah}]\subset[A_0]$, and $[B_{s+d}]$ is $[A_{2s+2d}]$. Now $[A_{2s}]$, which is $[A_{2|b|k}]$, is a subset of $[A_0]$ by 2:1, so $[A_{2s}]$, $[A_{2s+d}]$, $[A_{2s+2d}]$ are all subsets of $[A_0]$.

Suppose $[A_{js}]$, $[A_{js+d}]$, $[A_{js+2d}]$, $\cdots$, $[A_{js+jd}]$, $j<s/d$, the inductive hypothesis, are all subcollections of $[A_0]$. Then of course $[A_{(j+1)s}]$, $[A_{(j+1)s+d}]$, $[A_{(j+1)s+2d}]$, $\cdots$, $[A_{(j+1)s+jd}]$ are all subcollections of $[A_0]$ by 7:1 and the fact that $s$ is a multiple of $k$. Let $[B_0]$ be $[A_{js+jd}]$. Then $[B_0]\subset[A_0]$, so $[B_s]$ is a subset of $[A_0]$ because it is the set of the $|b|k$th derived sets of a subset of $[A_0]$, and also $[B_{s+d}]$ is a subset of $[A_0]$ because it is the set of the $ah$th derived sets of a subset of $[A_0]$. But $[B_{s+d}]=[A_{(j+1)s+(j+1)d}]$. Therefore all of the collections $[A_{(j+1)s}]$, $[A_{(j+1)s+d}]$, $\cdots$, $[A_{(j+1)s+(j+1)d}]$ are in $[A_0]$. The induction is now complete, and the required number $r$ is $s^2/d$. For $A$ is closed of grade $s$. Thus the collection of its sets of derived sets is a collection of $s$ sequences each of which is monotone decreasing. The particular sets $[A_r]$, $[A_{r+d}]$, $[A_{r+2d}]$, $\cdots$, $[A_{r+(s/d-1)d}]$ are all subcollections of $[A_0]$ and belong to $s/d$ different ones of these monotone sequences. All subsequent members of these special sequences are thus in $[A_0]$ and form the members of the collections $[A_{r+nd}]$.

Similarly in case (2) take $s=|a|h$ and $r=s^2/d$. It may be mentioned that in both cases $r$ is certainly an integer, for $d$ is a divisor of both $h$ and $k$, one or the other of which is a factor of $s$.

9:1 Theorem. *If $A$ is a set and $t$ is the least natural number for which there exists an integer $m$ such that $[A_{m+nt}]$, $n=0,1,2,\cdots$, is for all values of $n$ a subcollection of $[A_0]$, then there is a natural number $m_0$ divisible by $t$ such that each of $[A_{m_0+nt}]$, $n$ as above, is a subcollection of $[A_0]$.*

*Proof.* The set $A$ is obviously closed of grade $m$, as well as closed of grade $m+t$. If $m$ were not divisible by $t$, the greatest common divisor of $m$ and $m+t$ would be a natural number $d$ smaller than $t$, because any divisor of both $(m+t)$ and $(m)$ is a divisor of $(m+t)-(m)$. By 8:1 there would then be a natural number $r$ such that all of the collections $[A_{r+nd}]$ are composed of sets which belong to $[A_0]$. This contradicts hypothesis about the nature of $t$. As $m$ is divisible by $t$ it may be identified with the required number $m_0$. Thus each of $[A_{m_0+nt}]$ is a subcollection of $[A_0]$.

10:1 Theorem. *If $A$ is a set and $m$ and $t$ are natural numbers such that for every integer $n$ not less than $m$ the collection $[A_{nt}]$ is a subcollection of $[A_0]$, then for given natural number $q$ which is not less than $m$ each of the collections $[A_{(2q+i)t}]$, $i=1,2,\cdots$, is a subcollection of $[A_{qt}]$.*

*Proof.* Let $[B_0{}^i]$ represent $[A_{(q+i)t}]$. Then $[B_0{}^i]$ for each $i$ is a subcollection of $[A_0]$, and the set of all the $qt$th derived sets of the members of $[B_0{}^i]$ is a subset of $[A_{qt}]$; that is, $[B^i{}_{qt}] \subset [A_{qt}]$. But $[B^i{}_{qt}] = [A_{(q+i)t+qt}] = [A_{(2q+i)t}]$, so $[A_{(2q+i)t}] \subset [A_{qt}]$.

11:1 Theorem. *If $A$ is a set and $t$ is a positive integer for which $[A_{nt}]$, for every natural number $n$ greater than the definite integer $m-1$, is a subcollection of $[A_0]$, and $[B_j]$ and $[C_k]$ are monotone decreasing subsequences of the sequence $[A_{nt}]$, $n=m$, $m+1$, $m+2$, $\cdots$ then the subcollection of $[A_0]$ each of which is a member of each of $[B_j]$ is identical with the subcollection of $[A_0]$ each of which is a member of each of $[C_j]$.*

*Proof.* Let $[B_n]$ be any one, namely $[A_{n_b t}]$ of $[B_j]$. By 10:1 there is a natural number $m_b$ such that for every integer $i$ greater than $m_b$ each of the collections $[A_{it}]$ is a subcollection of $[A_{n_b t}]$. As $[C_k]$ is infinite, there must be a first of $[C_k]$ identical with one of the sequence $[A_{it}]$ for some $i$ greater than $m_b$, and thus a first of $[C_k]$ which is a subcollection of $[B_n] = [A_{n_b t}]$. In a similar way if $[C_n]$ is any one of $[C_k]$, there is one of $[A_{it}]$ in $[B_j]$ which is a subcollection of $[C_n]$.

In consequence any one of $[A_0]$ which belongs to all of $[B_j]$ belongs also to all of $[C_k]$, and vice versa.

12:1 Theorem. *If $A_1$, $A_2$, $A_3$, $\cdots$, $A_i$, $A_{i+1}$, $\cdots$ is a monotone decreasing sequence of sets each of which is closed of grade $n$ and if $A$ is the set of all points each of which belongs to all the sets in the sequence, then $A$ is also closed of grade $n$.*

*Proof.* Let $B$ be any subset of $A$. For each $i$, $B$ is then a subset of $A_i$. Furthermore since each of $A_i$ is closed of grade $n$, then the $n$th derived set of $B$ is a subset of each $A_i$. In consequence the $n$th derived set of $B$ belongs to each $A_i$ and thus to $A$. This means that $A$ is closed of grade $n$, for $B$ is any subset of $A$.

12:2 Corollary. *If $A_1$, $A_2$, $A_3$, $\cdots$, $A_i$, $A_i+1$, $\cdots$ is a monotone decreasing sequence of sets each of which is closed of a grade which is a divisor of $n$, and if $A$ is the set of all points each of which belongs to all the sets in the sequence, then $A$ is also closed of grade $n$.*

13:1 Theorem. *If $A_1$, $A_2$, $A_3$, $\cdots$, $A_i$, $A_{i+1}$, $\cdots$ is a sequence of sets each of which is closed of a grade which is a divisor of $n$, and if $A$ is the set of all points each of which belongs to all the sets in the sequence, then $A$ is closed of grade $n$.*

*Proof.* Let $B_1$ be $A_1$, $B_2$ be $B_1 \cdot A_2$, $B_3$ be $B_2 \cdot A_3$, $\cdots$, and in general $B_i$ be $B_{i-1} \cdot A_i$. Then the sequence $B_1$, $B_2$, $B_3$, $\cdots$, $B_i$, $\cdots$ is mono-

tone decreasing, and the set of all points each of which belongs to all the sets in this sequence is $A$. By easy adaptation of 4:1 each of $B_1, B_2, B_3, \cdots, B_n, \cdots$ is closed of grade $n$. By 12:1 then $A$ is closed of grade $n$ also.

14:1 *Definitions.* Two sets $A$ and $B$ are said to be ***compatible of grade k*** whenever a set $K$ which contains the $k$th derived sets of all of the subsets of $A$ and also the $k$th derived sets of all of the subsets of $B$, contains also all of the $k$th derived sets of all of the subsets of $A+B$. A class $[A_\alpha]$ of sets is compatible of grade $k$ whenever a set $K$ which contains the $k$th derived sets of all of the subsets of each of the members of $[A_\alpha]$ contains also the $k$th derived sets of all of the subsets of $\sum A_\alpha$.

14:2 Theorem. *If $A$ and $B$ are compatible of grade $k$, so are the three sets $A$, $B$, and $A+B$.*

*Proof.* If $K$ is a set containing the $k$th derived sets of all of the subsets of $A$, and of $B$, and of $(A+B)$; it contains therefore the $k$th derived sets of all of the subsets of $A+B+(A+B)=(A+B)$.

14:3 Theorem. *If $B$ is a subset of $A$, then $A$ and $B$ are compatible of grade $n$ where $n$ is any natural number whatever.*

*Proof.* Any set which contains the $k$th derived set of every one of the subsets of $A$, contains also the $k$th derived set of every one of the subsets of $B$, provided $B$ is a subset of $A$, and contains therefore the $k$th derived set of every one of the subsets of $A+B$, because $A+B$ is merely $A$.

14:4 Theorem. *The class of all of the subsets of the set $A$ is compatible of grade $n$ where $n$ is any natural number whatever.*

*Proof.* Let $k$ be any natural number and let $[A_k]$ be the class of the $k$th derived sets of all of the subsets, $[A_0]$, of $A$. Suppose that $K$ contains all of the members of $[A_k]$. Then $K$ contains the $k$th derived set of each one of $[A_0]$ by supposition. As $[A_0]$ includes $A$ and all of its subsets, then $K$ contains not only the $k$th derived sets of all of the subsets of each of $[A_k]$, as every subset of any one of $[A_k]$ is one of $[A_0]$, but also the $k$th derived sets of all of the subsets of $A$, which is the sum of the members of $[A_0]$.

15:1 Theorem. *If the set $A$ is closed of grade $h$ and the set $B$ is closed of grade $k$, and if $A$ and $B$ are compatible of grade $n$ where $n$ is a multiple of both $h$ and $k$, then $A+B$ is closed of grade $n$.*

*Proof.* By 2:1 $A$ is closed of grade $n$, that is, each member of the set $[A_n]$ of the $n$th derived sets of the set $[A_0]$ of all subsets of $A$ is itself a subset of $A$; and $B$ also is closed of grade $n$, that is, each member

of the set $[B_n]$ of the $n$th derived sets of the set $[B_0]$ of all subsets of $B$ is itself a subset of B. Thus $A+B$ contains the $n$th derived set of every subset of $A$ and the $n$th derived set of every subset of $B$. As $A$ and $B$ are compatible of grade $n$, then $A+B$ contains also the $n$th derived set of every subset of $A+B$. Because of 1:1 $A+B$ is therefore closed of grade $n$.

15:2 Corollary. *If A and B are compatible of grade k and each of them is closed of grade k, then their sum is closed of grade k.*

15:3 Corollary. *If a set of sets is compatible of grade k and each of the sets is closed of grade k, then the sum of all the sets in the set is closed of grade k.*

*Proof.* Owing to compatibility each subset of the sum of the sets has its $k$th derived set in the sum of the sets.

16:1 Theorem. *If the set A is closed of grade k, and for every natural number n each of the collections $[A_{nk}]$ of the nkth derived sets of the subsets of A is compatible of grade k and composed of members each of which itself is closed of grade k, then the set B of all points each of which belongs, for every value of n, to some member of the collections $[A_{nk}]$ is closed of grade k.*

*Proof.* For each $n$, $n$ a natural number, let $B_n$ be the set of all points each of which is contained in at least one of the collection $[A_{nk}]$. Then, for each $n$, $B_n$ is closed of grade $k$ by 15:3. Furthermore because of 6:1 for each $n$ $[A_{nk}] \supset [A_{(n+1)k}]$ so that $B_1, B_2, \cdots, B_n, \cdots$ is a monotone decreasing sequence of sets all closed of grade $k$. As $B$ is the set of all points each of which belongs to every one of the sequences, it is closed of grade $k$ by 13:1.

17:1 *Comments.* In the more commonplace spaces any finite number of sets is compatible of grade each natural number, but other collections of sets are in general not compatible at all. Owing to the highly arbitrary character of the law £, in any particular space, $S$ and £, the existence of aggregates of compatible sets would be unlikely. Upon similar grounds the occurrence of sets closed of a definite grade in a particular such space is also random. In any investigations where the aim is to construct a space comfortable to work with because of the familiar shape of its characteristics, it would be necessary to assign this shape by means of assumptions. One such assumption might be this: *Any finite collection of sets is compatible of grade any natural number.* In a space where there are any sets whatever which are closed of grade $k$, it would then follow from this assumption that the sum of any two of these is likewise closed of grade $k$.

Topics upon closed sets would also have to be given reality by means of suitably framed suppositions. Some light will be thrown below upon the form of these by a consideration of the role of the vacuous set in the space.

17:2 *Notation.* Let the vacuous set be $Z$. For each natural number $k$, let the $k$th derived set of $Z$ by $Z_k$.

18:1 Theorem. *If the set $A$ is closed of grade $k$, then the* $k$th *derived set $Z_k$ of the vacuous set $Z$ is in $A$.*

18:2 Theorem. *If the sets $A$ and $B$ are non-vacuous and both of them are closed of grade $k$, and if $Z_k$ is not $Z$, then $A$ and $B$ have points in common.*

*Proof.* Both $A$ and $B$ contain $Z_k$, because both contain $Z$.

18:3 Theorem. *If the sets $A$ and $B$ are non-vacuous, non-intersecting, and closed of grade $k$, then the* $k$th *derived set of $Z$ is $Z$.*

18:4 Theorem. *If there is no natural number $k$ such that $Z = Z_k$ and there are sets which are closed of finite grade, then any two such sets have points in common.*

*Proof.* If $A$ and $B$ are sets closed of finite grades $h$ and $k$ respectively then they are both closed of grade $hk$. As $A \supset Z$ then $A \supset Z_{hk}$; as $B \supset Z$ then $B \supset Z_{hk}$. Thus $A \cdot B \supset Z_{hk}$, a non-vacuous set.

18:5 Corollary. *All sets closed of the same finite grade $k$ contain the set*

$$\sum_{i=1}^{\infty} Z_{ki} = Z_k + Z_{2k} + \cdots + Z_{nk} + \cdots .$$

18:6 Theorem. *If no finite derived set of $Z$ is $Z$, the product of all those sets which are closed of grade $k$, $k$ finite, is a set $P^k$, which is non-vacuous and closed of grade $k$.*

*Proof.* The set $P^k$ is non-vacuous because it contains the non-vacuous set

$$\sum_{i=1}^{\infty} Z_{ki}.$$

Furthermore any subset of $P^k$ has its $k$th derived set in each of the sets which is closed of grade $k$, and therefore in their product $P^k$.

18:7 Theorem. *The sets $P^i$, where the natural number $i$ ranges over all indexes for which closed sets of order $i$ exist, and $P^i$ is the product of all sets closed of order $i$, have the following properties whenever $Z$ is not*

***a derived set of Z:*** (1) ***every closed set of finite grade contains at least one of them;*** (2) ***any finite collection of them has a non-vacuous product which includes a derived set of Z of some grade.***

18:8 *Example.* Space consists of the sequence of distinct points $a_1, a_2, a_3, \cdots, a_i, \cdots$; with derived relations as indicated by the arrows, and the additional provision that the derived set of each set is the sum of the derived sets of its points.

$$0 = a_0 \to a_1 \to a_2 \to a_3 \to \cdots \to a_n \to a_{n+1} \to \cdots .$$

Under these conditions:

$$\sum_{i=0}^{\infty} a_i$$

is closed of grade 1, is moreover the only set in space which is closed of grade 1, and therefore is $P^1$;

$$\sum_{i=0}^{\infty} a_{2i}$$

is closed of grade 2, is moreover the only set in space which is closed of grade 2, and therefore is $P^2$.

.. .. .. .. .. .. ..

$$\sum_{i=0}^{\infty} a_{hi}$$

is closed of grade $h$, is moreover the only set in space which is closed of grade $h$, and therefore is $P^h$;

.. .. .. .. .. .. ..

Any two of the sets closed of finite grade above have points in common, for instance $P^h$ and $P^k$ have in common $P^{hk}$. Any finite collection of these sets consists of members similarly with a point in common. But there is no point common to all of them. Because, regardless of $h$, the product of all of them does not contain $a_h$. In fact

$$P^{h+1} = \sum_{i=0}^{\infty} a_{i(h+1)}$$

does not contain $a_h$.

19:1 *Comments.* In case $Z$ has a vacuous derived set of some finite order the incidence relations of sets which are closed of some finite grade are more explicit. Suppose, to be definite, that $r$ is the smallest natural number for which $Z = Z_r$. Then the derived sets of $Z$ are the $r$ distinct sets $Z_1, Z_2, \cdots, Z_{r-1}, Z_r = Z$, and no others because

these are repeated indefinitely. Suppose also that $k$ is the smallest natural number such that the set $A$ is closed of grade $k$.

19:2 Theorem. *If $k$ is prime to $r$, $A$ contains each derived set of $Z$.*

*Proof.* When $k$ is prime to $r$, then the congruence $kx \equiv b \bmod r$, $b$ being any integer from 1 to $r-1$, has a positive integral solution $x_b$ corresponding to $b$. In consequence each of the derived sets of $Z$ is a subset of $A$; in particular $Z_b$ is a subset of $A$, because $A$ contains $Z$ and therefore contains $Z_{kx_b}$ which is identical with $Z_b$.

19:3 Theorem. *If $A$ and $B$ are closed of grades $k$ and $h$ respectively both of which are prime to $r$, then $A$ and $B$ have in common all of the derived sets of $Z$.*

19:4 Theorem. *If $k$ and $r$ have the greatest common divisor $d$, an integer between 1 and $r$, then $A$ contains the derived sets of $Z$ which are of the form $Z_{bd}$, $b$ being any integer from 1 to $r/d$.*

*Proof.* When $k$ and $r$ have the greatest common divisor $d$, then the congruence $kx \equiv bd \bmod r$ has a solution for all values of $b$ from 1 to $r/d-1$. Since $Z_{kx}$ is a subset of $A$, for each value of $x$, then $A$ contains $Z_{bd}$ for each of the values of $b$ mentioned above.

19:5 Theorem. *If $A$ and $B$ are closed of grades $k$ and $h$ respectively both of which have the greatest common divisor $d$ with $r$, then $A$ and $B$ have in common the set*

$$\sum_{b=1}^{r/d} Z_{bd}.$$

19:6 Theorem. *The product of all those sets which are closed of grade $k$, $k$ not divisible by $r$, is a set which is non-vacuous and closed of grade $k$.*

*Proof.* That the product is non-vacuous follows by means of reasoning as in 19:2 and 19:4. If $B$ is any subset of the product then its $k$th derived set is in each of the members of the product and thus in the product.

19:7 Theorem. *There is no non-vacuous set necessarily common to two set which are closed of grade $r$, and not closed of any positive natural grade smaller than $r$.*

20:1 *Comments.* Owing to considerations similar to those just above, in case the derived set of $Z$ is $Z$ itself, there is no non-vacuous set necessarily common to two sets which are closed of some finite grades. As a detail in the construction of a space comfortable to work with it would thus be convenient to make the assumption: $Z = Z_1$.

# On Certain Limits

By Alfred Brauer
*University of North Carolina*

In this Magazine, Vol. 17, (1943), p. 234, W. E. Byrne put the following question:

There seems to be a lack of variety of problems of the type

$$\lim_{x\to 0} [f(x)]^{g(x)} \quad \text{where} \quad \lim_{x\to 0} f(x)=0 \quad \text{and} \quad \lim_{x\to 0} g(x)=0.$$

Practically all that are given in texts have the answer 1.

To answer this question let us prove the following theorem:

Theorem. Let $k$ and $l$ be positive integers and $m=k+l-2$. Suppose that the functions $f(x)$ and $g(x)$ and their first $m+2$ derivatives exist and are bounded in the interval $0<x\leqq h$; let $f(x)$ be positive for $0<x\leqq h$. Moreover, suppose that

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} f'(x)=\cdots=\lim_{x\to 0} f^{(k-1)}(x)=0, \tag{1}$$

$$\lim_{x\to 0} f^{(k)}(x)\neq 0; \tag{2}$$

$$\lim_{x\to 0} g(x)=\lim_{x\to 0} g'(x)=\cdots=\lim_{x\to 0} g^{(l-1)}(x)=0, \tag{3}$$

$$\lim_{x\to 0} g^{(l)}(x)\neq 0. \tag{4}$$

Then

$$\lim_{x\to 0} [f(x)]^{g(x)}=1.$$

Proof. Set

$$\left.\begin{cases} N_\mu=\lim\limits_{x\to+0} \dfrac{d^\mu}{dx^\mu}[f'(x)g^2(x)] \\[2ex] D_\mu=\lim\limits_{x\to+0} \dfrac{d^\mu}{dx^\mu}[f(x)g'(x)] \end{cases}\right\} (\mu=0,1,\cdots, m+1). \tag{5}$$

We want to prove that

$$N_0=N_1=\cdots=N_{m+1}=0, \tag{6}$$

$$D_0=D_1=\cdots=D_m=0, \tag{7}$$

but

$$D_{m+1}\neq 0. \tag{8}$$

It follows from Leibniz's theorem for the derivatives of the product $f(x)g'(x)$ that

$$D_\mu = \sum_{\kappa=0}^{\mu} \lim_{x\to+0}\left[\binom{\mu}{\kappa} f^{(\kappa)}(x)g^{(\mu-\kappa+1)}(x)\right] \quad (\mu=0,1,\cdots,m+1).$$

For $\mu \leqq m$ each term of the sum contains either a factor $f^{(\kappa)}(x)$ with $\kappa \leqq k-1$ or a factor $g^{(\lambda)}(x)$ with $\lambda \leqq l-1$ since for $\kappa \geqq k$

$$\mu-\kappa+1 \leqq m-k+1 = k+l-2-k+1 = l-1.$$

Since the other factors are bounded, each term of $D_\mu$ has the limit 0 for $\mu \leqq m$ by (1) and (3). All the terms of $D_{m+1}$ have the limit 0, except the term

$$\binom{m+1}{k} f^{(k)}(x)g^{(m+1-k+1)}(x) = \binom{m+1}{k} f^k(x)g^l(x)$$

which has a limit different from 0 by (2) and (4). This proves (7) and (8).

Similarly,

$$N_\mu = \sum_{\substack{\kappa,\lambda \\ \kappa+\lambda\leqq\mu}} \lim_{x\to+0}\left[\frac{\mu!}{\kappa!\,\lambda!\,(\mu-\kappa-\lambda)!} f^{(\kappa+1)}(x)g^{(\lambda)}(x)g^{(\mu-\kappa-\lambda)}(x)\right].$$

For $\mu \leqq m+1$ each term contains either a factor $f^{(\kappa+1)}(x)$ with $\kappa \leqq k-2$ or a factor $g^{(\rho)}(x)$ with $\rho \leqq l-1$ since for $\kappa \geqq k-1$ and $\lambda \geqq l$

$$\mu-\kappa-\lambda \leqq m+1-k+1-l = k+l-2+2-k-l = 0.$$

This proves (6)

From (5), (6), (7), and (8) it follows that $N_{m+1}/D_{m+1}$ exists and that

$$(9) \qquad \lim_{x\to+0} \frac{f'(x)g^2(x)}{f(x)g'(x)} = \lim_{x\to+0} \frac{N_0}{D_0}$$

$$= \lim_{x\to+0} \frac{N_1}{D_1} = \cdots = \lim_{x\to+0} \frac{N_{m+1}}{D_{m+1}} = 0.$$

Hence there is an interval $(0, h_1)$ inside which $g(x)$ does not vanish),* Therefore we have

$$\lim_{x\to+0} [g(x)\cdot\log f(x)] = \lim_{x\to+0} \frac{\log f(x)}{1/g(x)}$$

$$= \lim_{x\to+0} \frac{f'(x)/f(x)}{-g'(x)/g^2(x)} = -\lim_{x\to+0} \frac{f'(x)g^2(x)}{f(x)g'(x)},$$

*See e. g. G. H. Hardy, *A Course of Pure Mathematics*, 7th ed., Cambridge 1938, pp. 294-295.

hence by (9)
$$\lim_{x\to+0} [g(x)\cdot\log f(x)] = 0,$$
$$\lim_{x\to+0} [f(x)]^{g(x)} = 1.$$

and the theorem is proved.

From this theorem we obtain the following result:

Suppose that $f(x) > 0$ and $g(x) \neq 0$ for $0 < x \leqq h$. If all the limits

$$\lim_{x\to+0} f^{(v)}(x) = a_v \quad \text{and} \quad \lim_{x\to+0} g^{(v)}(x) = b_v \qquad (v = 0,1,2,\cdots)$$

exist, and if $a_0 = b_0 = 0$, then

$$\lim_{x\to+0} [f(x)]^{g(x)} \neq 1$$

only if either all the limits $a_v$ or all the limits $b_v$ vanish.

On the other hand $\lim_{x\to+0} [f(x)]^{g(x)}$ can be every positive number. If $0 < f(x) < 1$ for $0 < x \leqq h$ and if

$$\lim_{x\to+0} f(x) = 0, \quad \lim_{x\to+0} F(x) = 1, \quad g(x) = \frac{cF(x)}{\log f(x)}$$

where $c$ is a constant, then $\lim_{x\to+0} g(x) = 0$, and

$$\lim_{x\to+0} [f(x)]^{g(x)} = \lim_{x\to+0} e^{g(x)\log f(x)} = \lim_{x\to+0} e^{cF(x)} = e^c.$$

For instance,

$$\lim_{x\to+0} (e^{-(1/x)})^{-c\sin x} = \lim_{x\to+0} e^{(c\sin x)/x} = e^c$$
$$\lim_{x\to+0} x^{c/\log x+x} = \lim_{x\to+0} e^{c\log x/(\log x+x)} = e^c.$$

# *Humanism and History of Mathematics*

*Edited by*
G. WALDO DUNNINGTON and A. W. RICHESON

## A Seventh Lesson in the History of Mathematics

By G. A. MILLER
*University of Illinois*

16. *The Triangle.* The plane triangle is the only rectilinear figure which is completely determined by its sides and hence it is fundamental in the construction of buildings as well as in the determination of areas. One of the most fundamental invariants connected with it is that the sum of its angles is independent of its shape, being always equal to two right angles. It is not known who first proved this theorem or even who first formulated it. Like the first proof of the fact that the square root of 2 is not a rational number it is unlikely that it can ever be determined when it first became known but this was before the appearance of the *Elements* of Euclid, and the ancient Greeks were very much interested in it. Both of these theorems are of great theoretic interest and hence they portray the spirit of Greek mathematics rather than that of pre-Grecian mathematics, since the latter contains few general theorems in an explicit form.

The determination of the area of the plane triangle is a more practical matter and was naturally of much interest to the pre-Grecian mathematicians. The ancient Egyptians made frequent use of a rule which is only approximately true, viz., that the area of a plane triangle is equal to one-half of the product of two of its sides. It was not at first observed in the study of the ancient Egyptian mathematics that they also knew a correct rule for finding this area, and this oversight led to a large number of inaccurate publications relating to the ancient Egyptian mathematics.* It should be observed that the shape of the unit of measure may be involved. The ancient Babylonians knew the so-called Pythagorean theorem, and their early mathematical

*For instance, the matter is incorrectly stated in Smith's *History of Mathematics*, Vol. II, p. 290.

knowledge was on a much higher level than that of the ancient Egypttians. They used the so-called Pythagorean theorem at least as early as the close of the third millennium B. C. and they are known to have continued to use it for about two thousand years.

While some of the pre-Grecian mathematicians including some of the ancient Egyptians, knew that the area of a plane triangle is equal to the product of one-half of the base into the altitude the ancient Greeks seem to have been the first to express the area of a general plane triangle in terms of its three sides $a$, $b$, $c$ by means of a rule which is equivalent to the formula

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where $s$ represents one-half of the perimeter of the triangle. This rule is remarkable on account of its age and the formula is commonly called Heron's formula. It appears in the works of Heron but it was probably known earlier and it has been suggested that it may be due to Archimedes (287-212 B. C.). It does not appear in Euclid's *Elements*, which were written shortly before the time of Archimedes and embody many (but not all) of the important mathematical results found by the predecessors of Euclid. In fact, these *Elements* relate mainly to theoretical considerations in regard to arithmetic, algebra and geometry. They are not confined to geometry as has often been assumed, but they emphasize the geometric forms of statements.

It has frequently been stated that the ancient Egyptian surveyors used the plane triangle whose sides are in the proportion 3, 4, 5 to construct right angles. On page 64 of E. T. Bell's *Development of Mathematics* (1940), this view is expressed as follows: "The Egyptian 'rope stretchers' laid out right angles for the orientation of buildings by means of a triangle of sides 3, 4, 5. A string of length $3+4+5$ was marked or knotted at the points 3, 4. With this and three pegs a right angled triangle was obtained in an obvious way." It had frequently been pointed out before the appearance of this book that there is now no extant definite evidence that the ancient Egyptians constructed right angles in this way. It has been made plausible, however, that they constructed right angles by connecting the middle of the base of an isosceles triangle with the vertex of this triangle. In some of the Egyptian pyramids the four base angles are very close to right angles.

The theorem that the angles opposite the equal sides of an isosceles triangle are equal to each other has often been credited to Thales, but it is probably much older. This is also true of other theorems which used to be credited to Thales. In fact, we know now practically nothing in regard to the geometrical contributions of Thales and

Pythagoras. Mathematics among the ancient Greeks was then much less advanced, judging from many discoveries which seem to be due to later authors, than the writers on the history of Greek mathematics used to assume. In particular, it does not seem likely that Pythagoras was able to prove the theorem which is known by his name and hence the statement that he sacrificed a hundred oxen (or even one ox) when he discovered this theorem has been discredited by practically all recent writers on the history of Greek mathematics. The correct use of the Pythagorean theorem seems to be more than a thousand years older than its satisfactory proof. The latter was probably due to the Greek successors of Pythagoras and it appears in Euclid's *Elements*, Book I.

A plane triangle is now often defined as a geometrical figure composed of three points, called the vertices, not lying in a straight line, and the three straight line segments joining them, called the sides. On the contrary, the Greek theoretical mathematicians, including Euclid, usually understood by their geometrical figures areas but not line and point configurations. It may be noted here that in the first edition of the *Mathematics Dictionary* by Glenn James and Robert C. James (1942), the circle and the sphere are defined as a curve and a surface respectively, while in the revised edition (1943) they are also defined as a surface and a volume respectively. The question naturally arises to what extent it is desirable to use the same term in mathematics for widely different concepts. A dictionary and a history should clearly aim to give the actual facts irrespective of whether they were used in a justifiable way. Some ambiguities have always been tolerated in mathematics notwithstanding the opposition thereto.

Figures of triangles appear in the writings of the ancient Egyptians but they are represented therein as lying on their sides instead of standing on their bases. A triangle standing on its base in their extant writings represents a pyramid or a cone. A large number of theorems relating to the plane triangle appear in the *Elements* of Euclid but the corresponding theorems relating to the spherical triangles were omitted therein and were given by Menelaus several centuries later (about 100 A. D.), who modeled his treatment on Euclid's treatment of plane figures. He proved for instance, that the sum of the angles of a spherical triangle is not a constant, as in the case of the plane triangle, but always exceeds two right angles. When two plane triangles have their sides respectively equal to each other they can be superimposed but this is evidently not necessarily true as regards two spherical triangles even on the same sphere.

The ancient Greeks were much interested in the plane triangle whose angles are equal to 30°, 60°, and 90° respectively. This triangle

was said by Plato (429-348) to have the most beautiful form for unknown reasons. Its sides are to each other as 1, 2, $\sqrt{3}$, and Plato observed that the equilateral triangle, the square, and the regular hexagon can be divided so as to form two kinds of elementary right triangles, viz., those having two of the sides equal to each other and those in which the acute angles are 30° and 60° respectively. By connecting the center of a square with its vertices there result four right triangles whose shorter sides are equal to the radius of the circumscribed circle and by connecting a vertex of an equilateral triangle with the middle of the opposite side there result two right triangles whose acute angles are 30° and 60° respectively. The regular hexagon can obviously be divided into regular triangles and hence, also into right triangles whose acute angles are 30° and 60°.

Number triads like 3,4,5 which are equal to the measures of the three sides of a plane right triangle are often called Pythagorean numbers, but many of them were known long before the days of Pythagoras. In ancient Babylonian mathematics at least the following four such triads were noted: 3,4,5; 8,15,17; 5,12,13; 20,21,29. A larger number of such triads have been noted in the ancient Indian mathematics. The latter sometimes include irrational numbers such as 1, $\sqrt{2}$, $\sqrt{3}$; 1,3,$\sqrt{10}$; and 2,6,$\sqrt{40}$. Triads appear also in the ancient Egyptian mathematics but no proof of the general so-called Pythagorean theorem has yet been found in the ancient literature of these peoples. The ancient Hindus expressed the general Pythagorean theorem with respect to the diagonal of a rectangle and the ancient Babylonians used this theorem with respect to the diagonal. *Many other evidences tend to show that the ancient Indian mathematics was influenced by earlier work of the Babylonians and the Greeks.* It has often been unduly praised as a result of the writings of Birk and others, and the tendencies to exaggerate in historical writings.

The plane right triangle with the sides equal to 3,4,5 obviously has 6 for its area. According to the French mathematician V. A. Lebesgue (1791-1875) this is the only plane right triangle whose sides and area form an arithmetic progression. The ancient Greeks interested themselves in the question of finding plane right triangles having integers for their sides. The Pythagoreans found that the sides of such a triangle may have the general forms represented by $2q+1$, $2q^2+2q$, and $2q^2+2q+1$ respectively, and Plato gave the corresponding forms as $m^2-1$, $2m$, and $m^2+1$. In his *Elements* Euclid gave a more general formula which includes those already noted as special cases. Such right triangles are sometimes called *Pythagorean triangles* and by adjoining two such right triangles with equal sides we obtain an oblique triangle whose sides are integers. Heron gave an example of such a

triangle with the sides 13, 14, 15, and the area 84. The name *Heron triangles* is sometimes used to denote this general type of triangles. This type of mathematics has not been very fruitful.

It is remarkable that in Euclid's *Elements* we find only three of the four common theorems relating to the congruence of two plane triangles, viz., those which affirm that two such triangles are congruent whenever the three sides of the one are respectively equal to the three sides of the other, when a side and the two adjacent angles of the one are equal to the corresponding parts of the other, and when two sides and the included angle of the one are equal to the corresponding parts of the other. The fourth common congruence theorem, viz., when two sides and an angle opposite one of these sides are equal to the corresponding parts of another triangle, does not appear in the works on elementary geometry before the beginning of the eighteenth century. As is well known this case may give rise to two possible triangles but it seems difficult to explain why it was omitted in Euclid's *Elements*, especially since he treated the corresponding case for similar plane triangles. What is perhaps of greater historical interest is that this fourth congruence theorem made its appearance so recently in the works on elementary geometry.

It is interesting to observe that in Euclid's *Elements* no stress is laid on what are now commonly known as the *remarkable points* of the plane triangle. In considering the circles inscribed in such a triangle and circumscribed about it Euclid noted that the center of the former circle is the point of intersection of the two lines which bisect two of the angles of the triangle without explicitly noting that the three bisectors of the angles of the triangle are concurrent. Similarly, in noting the circumscribing circle of a given plane triangle he merely noted that the perpendiculars erected at the midpoints of two of the sides of a given triangle, intersect in the center of the circumscribed circle, without noting explicitly that the perpendicular erected at the midpoint of the third side of this triangle passes through the same point. That is, he did not note that we have here two remarkable points of the given triangle, as we now say.

The third remarkable point of the plane triangle, its center of gravity, is not even mentioned by Euclid. This point was noted later by Archimedes (287-212), who knew that it is the point of intersection of the three lines which are determined by the vertices of this triangle and the midpoints of the opposite sides, but even he did not emphasize this fact as an interesting geometric theorem. It was explicitly noted later by Heron, and in medieval times it was noted by the Italian writer, Leonardo, but it received very little attention before the nineteenth century. The fourth remarkable point of the plane triangle is the

point of intersection of the three lines drawn through the vertices of the triangle perpendicular to the opposite sides. While Archimedes knew the theorem that these three lines are concurrent he did not emphasize it as a special theorem and it received very little attention before the eighteenth century. In fact, during medieval times mathematicians usually paid little attention to the then known geometrical facts which do not appear in the *Elements* of Euclid. They seldom attempted to extend mathematical knowledge beyond what had been transmitted to them from the ancients.

During the nineteenth century a very large number of articles appeared which were devoted to the study of the remarkable points of the triangle as may be seen by consulting the *Royal Society Index* 1800-1900, volume I, pages 445-7, (1908). In fact, there was then developed what has been called a newer geometry of the triangles and the number of known remarkable points of the triangle was greatly increased. A French author, E. Lemoine (1840-1912), wrote extensively on this subject and he was called the founder of the geometry of the triangle in the former French journal *L'Intermédaire des Mathématiciens,* volume 19, page 49 (1912). The number of new facts relating to the triangle which have been listed since the beginning of the nineteenth century is very large and the subject is probably not yet exhausted, but may possibly be too special.

It was noted above that the ancients found the area of a plane triangle, often approximately, by the use of three different rules, viz., one-half the product of two sides, the product of the base into one-half of the altitude, and the rule based on what is commonly known (probably incorrectly) as Heron's formula. While the first of these rules is fundamentally incorrect it should be emphasized that when the two sides in question are nearly at right angles to each other the result thus obtained is a very close approximation to the correct result. The other two rules usually depend on the extraction of the square root since the altitude cannot be directly measured in most cases. An approximation rule for the extraction of the square root appeared already in the writings of the ancient Babylonians, viz., when a number is written in the form $a^2+b$, where $b$ is relatively small, its approximate square root is $a+(b/2a)$. This rule was widely used later.

A much more recent formula for finding the area of the plane triangle whose two sides $a$, $b$ include the angle is $\frac{1}{2}ab \sin C$. From this formula it follows directly as is otherwise evident that the maximum area of a triangle with two sides $a$ and $b$ is $ab/2$. The formula $\frac{1}{2}ab \sin C$ appears in a somewhat different form in the influential trigonometry of Snell (1627). The point to be emphasized in this connection is that it relates the finding of the area of a plane triangle to the large subject

plane trigonometry which also has an extensive history. The vast number of tables giving the approximate values of the trigonometric functions are thus closely relating to the study of the plane triangle. As might be expected this geometric figure has exerted an influence on the development of mathematics for more than four thousand years and is related to various extensions of the number concept. It helped to inspire both exact mathematics and approximation mathematics during all this time.

The study of the spherical triangle, largely inspired by astronomy, is naturally more recent than that of the plane triangle. *Among the ancient Babylonians the pure mathematical developments extend at least a thousand years farther back than their calculations in astronomy* and hence the study of the spherical triangle seems to have received little attention before the times of the ancient Greeks.* It was noted above that in Euclid's *Elements* there does not even appear the theorem that the sum of the angles of a spherical triangle always exceeds two right angles. It was Archimedes who first announced many of the fundamental theorems relating to the sphere such as that its area is equal to four great circles. A fundamental theorem in Euclid's *Elements* about the sphere is that the volumes of spheres are to each other as the cubes of their dimaeters.

The clear concept of a spherical triangle appears for the first time in the work of Menelaus to which we referred above but the important concept of the polar triangle of a spherical triangle which exhibits an important duality in the study of this triangle seems to have been clearly noted for the first time in the work of F. Vieta (1540-1603), although it had been used earlier by the well known Arabian mathematician Nasir Eddin (1201-1274). A theorem giving the area of a spherical triangle seems to have been noted for the first time by T. Harriot (1603) and was proved by A. Girard in 1629. The fact that the sum of the angles of a spherical triangle is less than six right angles was noted in 1558 by F. Maurolico. Menelaus had only proved that this sum is always greater than two right angles. Since the spherical triangle is so intimately related to astronomy it may appear strange that these theorems were first proved so recently.

An article of about 150 pages relating to the spherical triangle was published by E. Study in the *Leipziger Abhandlungen*, volume 20 (1893). In this article the subject is viewed from an advanced standpoint and it is stated here that only by group theory considerations is it possible to gain a satisfactory insight into the concept of the spherical triangle. The well known Swiss mathematician, L. Euler (1707-1783)

*Cf. O. Neugebauer, *Geschichte der Antiken Mathematischen Wissenschaften*, p. 105 (1934).

called attention to the fact that when the radius of the sphere is assumed to be infinite the formulas of the spherical triangle reduce to formulas of the plane triangle. Hence the spherical triangle may be regarded as more general than the plane triangle and the study of the latter assumes additional interest when it is considered from this point of view. This has been done by various writers, including J. H. Lambert (1728-1777), who was a contemporary of L. Euler and first proved that $\pi$ is an irrational number.

In Euclid's *Elements*, Book I, plane triangles are sometimes called trilateral figures, and they are divided into equilateral triangles, isosceles triangles, scalene triangles, right-angled triangles, obtuse-angled triangles, and acute-angled triangles. Proclus (410-485) gave as a reason for the dual classification according to the sides and angles that not every triangle is also trilateral. He spoke of the four-sided triangle as one of the paradoxes in geometry. This triangle is really a plane quadrilateral with a re-entrant angle. The idea of calling it a triangle was based on the assumption that an angle could not be larger than 180°. This so-called triangle was also said to be barb-like. The first proposition of Euclid's *Elements* is devoted to the equilateral triangle and this triangle naturally received special attention on account of its symmetry. While the term triangle was commonly used by the ancient Greeks in the general sense they commonly restricted the term quadrangle to denote a square. This was done in the *Elements* of Euclid and hence became a common practice.

Euclid's *Elements* contain the theorem that in two equiangular plane triangles the sides about the equal angles are proportional but this theorem is much older and can probably never be traced to its source. It is connected with the measure of the height of an object from the length of its shadow and it has been suggested that Thales discovered it, but this is probably a myth. Euclid's *Elements* also contain the theorem that similar plane triangles are to each other as the squares of their corresponding sides. This theorem was foreshadowed by many early developments of the ancient Babylonians and the ancient Egyptians, including the assumption on the part of the latter that the area of a circle is equal to that of a square on 8/9 of the diameter of the circle. It was sometimes incorrectly assumed by the ancients that the areas of similar figures have the same ratio to each other as their perimeters instead of the square, thereof.

On a sphere two equiangular triangles have the same area and are either equal or symmetrical. Hence *mutually equiangular triangles on a sphere have a much less extensive history than those in the plane.* An important triangle on the sphere is the birectangular triangle whose third angle is one degree. The area of this is known as a spherical

degree. The French mathematician, A. M. Legendre (1752-1853) called the area of this triangle a unit area and this was also done by many of the later writers on the subject. The area of any triangle on a sphere can thereby be expressed as its spherical excess. That is, as the excess of its angles over two right angles. Both the 720 birectangular triangles whose separate areas are one spherical degree and the 8 trirectangular triangles into which the sphere can be divided have been largely used in modern times as triangular units of measure of portions of the surface of the sphere. Hence they are of special interest. It should be noted that the ancient Greeks became interested in the spherical triangle after their most active mathematical period.

The successive sums of the first $n$ natural numbers, viz., $1, 3, 6, 10 \cdots$ are known as triangular numbers and these numbers can obviously be arranged in the form of equilateral triangles. The early Greeks were interested in such numbers as well as in the more general series of numbers known as figurate numbers. Since the $n$th triangular number is equal to $n(n+1)/2$ it was later assumed by Adelbold, bishop of Utrecht, that this formula represented the area of an equilateral triangle of side $n$, and hence that the area of such a triangle whose side is 7 would be 28, while the rule that the area of a triangle is one-half of the base into the altitude led to a different result. A letter on this subject which Adelbold wrote to Gerbert, who became Pope in 999, is still extant and is evidence of the general lack of mathematical knowledge in Europe at that time. It is however interesting to observe that the Pope and a bishop were taking an interest in such a simple mathematical question relating to the plane triangle, even in the Dark Ages.

The group of movements of the plane equilateral triangle is the symmetric group of order 6 and the group of movements of the spherical equilateral triangle is the cyclic group of order 3 while the group of movements of the plane isosceles triangle is the group of order 2. All the other triangles have the identity for their group of movements. Although the first proposition of Euclid's *Elements* is devoted to the equilateral plane triangle nothing is said therein of its group of movements, which is the non-abelian group of smallest order and is illustrated, for instance, in the beginning of the textbook on *Finite Groups* by Miller, Blichfeldt, Dickson, (1916) by the movements of the plane equilateral triangle. Since these movements constitute a non-abelian group they emphasize the historical fact that the term *commutative* was not used in the mathematical literature before 1814 when F. J. Servois introduced the term. This throws light on all of the earlier mathematical work and hence deserves emphasis.

The ancients did very little in regard to finding additional parts, or additional facts, with respect to a triangle from the given facts. The main exception is that they found the area of a plane triangle by means of certain given parts, and, in the case of the plane right triangle, they found a third side when two sides were given. The determination of the approximate values of unknown parts of a triangle from a sufficient number of given parts is mainly a problem of modern times whose solution became feasible as a result of extensive tables, including logarithmic tables. The latter were made largely available during the seventeenth and eighteenth centuries and have become a prominent feature of approximate work in mathematics. More recently calculating machines have partly replaced logarithmic tables. Although the triangle belongs to geometry its study reaches into various other branches of mathematics and emphasizes the intrinsic unity of these branches. Evidences of this unity are fundamental in the history of mathematics and furnish its main charms for many readers.

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## *CLIFFORD B. UPTON, CHAIRMAN OF THE BOARD AMERICAN BOOK COMPANY*

Professor Clifford B. Upton, for many years Professor of Mathematics, Teachers College, Columbia University, was elected Chairman of the Board of Directors of American Book Company at its last meeting, effective November 1, 1943. He has been a member of the Board for many years, and has always taken an active part in the affairs of the Company.

Mr. Upton succeeds Mr. A. Victor Barnes who had asked the Board to accept his resignation as Chairman in view of his advancing years. Mr. Barnes has been associated with the American Book Company for nearly fifty years in various official capacities, has been a member of the Board, and has served as its Chairman. With the Company in such excellent condition and its affairs so well in hand, Mr. Barnes felt he would like to have more time to devote to his personal interests. He will remain on on the Board as a Director, and will continue to give the Company the benefit of his long experience.

Professor Clifford B. Upton brings to his new association qualifications and experience which assure a contribution of importance to the business of publishing school and college textbooks. His broad training and extensive knowledge of the problems of education, his years of experience in the training of teachers, and his extended acquaintance among school people everywhere, give Mr. Upton especial equipment for his new duties. Moreover, he has had unusual opportunity to put his abilities to practical use in the preparation and publication of many successful books of his own authorship. The American Book Company is to be congratulated on its success in meeting this important situation.

# Mathematics in a World at War—A Challenge to Mathematics Teachers

By JAMES H. ZANT
*Oklahoma Agricultural and Mechanical College*

During the last few months, mathematics has gained a position in public esteem which was never dreamed of by its most ardent supporters of a few years ago. The mathematicians of the world can take little credit for this gain in popularity, however. It has been due to the needs created by the emergency of a world at war, rather than to any effort on our own part. Whether or not we take advantage of the opportunity which has been handed to us does depend very much on our own efforts and thinking during the immediate future.

The importance of mathematics in the present emergency need not be discussed here. It should be familiar to any one who reads these lines. The effect the emergency is having on mathematics and mathematics teaching is most important and may well have a far reaching significance. Many of the mathematics courses now being taught in colleges and secondary schools are attempts to meet definite conditions in industry and in the armed services. These courses propose to teach the student the mathematics he will need to do the things he will have to do to succeed in a definite trade or in a branch of the Army or Navy. This is often done to the entire neglect of any of the meanings of the subject or any of the surrounding skills and knowledge which the better workman might need for a job a little out of the ordinary. This is sometimes justified by saying that time is valuable and therefore cannot be used for things of doubtful value. Certainly, if a soldier has only a limited amount of time for his training, important skills which may be a matter of life or death to him in his job cannot be neglected while we go into meanings which may be of no use to him whatever.

Some of the characteristics of these new courses may be summarized as follows:

1. They propose to teach those things, and only those things, which the student is sure to need in his future activity.
2. They propose to do this without any reference to tradition or what the traditional teacher thinks should be included in a "good" mathematics course.
3. They do not propose to waste time on meanings and cultural items whose value cannot be definitely demonstrated.
4. They do not propose to include a lot of extra material which professors of mathematics think will be good for the student.
5. They often fail to include enough of meanings and knowledge above the bare essentials to enable a student to go beyond what the ordinary worker or soldier is expected to do. This may not seem to be a good pattern for after the war teaching in the field of mathematics, but one must admit that for many of the students who try to go to college, it has its good points. Whether we like it or not, it may well be the pattern of much of our education, both secondary and collegiate, after the war is over. Certainly it is a problem which cannot be met by merely ignoring the facts.

One could probably say that mathematics and mathematics teachers have effectively met the emergency created by the war, particularly with regard to:

1. The amount of research and store of knowledge which has been usuable for both industry and war.
2. The personnel for leadership in further research needed in industry and also in the armed forces.
3. The personnel necessary to adapt the store of knowledge to Army and Navy problems and the officers to train the men to use it properly.
4. The trained leaders necessary to reorganize the teaching of mathematics in the colleges to fit the needs of men in the service and those soon to be in the service.

The emergency has not, however, been adequately met with respect to:

1. The knowledge and skills the student possessed on entering the Army and Navy. After studying mathematics for approximately ten years in the public schools and often in college, the

average soldier is still unable to add correctly a simple column of figures.

2. Older people as well as young ones who entered industry revealed the same lack of fundamental knowledge and skills, and most of them have to be retrained in essential mathematics before they are usable.
3. Mathematicians as a group have not been trained to organize their subject matter in such a way as to provide the needed skills and knowledge for a particular job which the student must learn to perform. That is, too many of us would still teach any student a course in mathematics, or preferably several courses, and hope that this would prepare him for the job at hand.

What of the future of mathematics? First, we should not be foolish enough to suppose that because mathematics has been able to give good service during the war and because our well-trained mathematics students are in demand, that the good reputation of mathematics will continue in peace times unless we change our teaching to fit the conditions, whatever they may be. One great danger, I think, will be in assuming that we can pick up where we left off in December, 1941, and do things as we were doing them at that time. No one knows what problems are going to face us after the war, but the best thinkers seem to be sure that they will be basically different from the past. Mathematics will have a head start at that time. It enjoys a reputation of being a valuable body of knowledge which is highly useful to a worker or a soldier in almost any field. For the time being, people are willing to overlook the fact we have been able to train only a small per cent of our total number of students so that they can perform these essential tasks. However, in peace times, we cannot expect this to continue. We must be able to produce results from our teaching, or we will lose all that we have gained.

To do this, we must not only teach better, but we must also make what we teach fit the conditions at hand. In engineering education we have always taken two or more years to teach the courses in engineering mathematics. Now the Army in the Army Specialized Training Program is doing this in a period of 48 weeks of actual classroom instruction. Other subjects of the engineering program are being speeded up in a similar fashion. While this may not be good engineering education, as our engineering teachers keep insisting, it may be good enough to do the jobs that most engineers have been doing in the past. Or, what is more important, it may be good enough for much of the engineering work of the future. If the public gets the idea that an

engineer may be trained in two years instead of four or five, it may insist on the colleges doing it. At any rate, the college can't ignore such facts, if it is to retain the prestige it has won during the present emergency.

Mathematics then can look forward to a healthful future. But its maximum service will be dependent on how it meets the changing conditions which are almost surely in store for us. If we look at each new situation that presents itself with unprejudiced eyes, and if we are determined to serve our students to the best of our ability, there is no reason why the subject should not continue to be, as it has been in the past, the greatest single factor in the progress of our civilization.

# Fractional Indices, Exponents, and Powers

By Carl B. Boyer
*Brooklyn College*

The word *power* is defined by Webster as "The product arising from the continued multiplication of a number into itself." The word exponent is taken to mean "A symbol written above another symbol and on the right, denoting how many times the latter is repeated as a factor."[1] However, so loosely have these terms come to be used that statements inconsistent with the above definitions can be found not only in Webster but also in almost any good elementary textbook or history of mathematics. For example, the word "logarithm" is defined as "The exponent of that power of a fixed number (called the *base*) which equals a given number (called the *antilogarithm*)."[2] In terms of the previous definitions this is sheer nonsense. Only very exceptionally do the logarithms of numbers turn out to be positive and integral, or even rational. Hence the exponent in this case usually designates not a repeated multiplication but a sequence of operations: raising to powers, finding principal roots, determining a limit, and then perhaps taking a reciprocal. Meticulous authors point out that in this case the meaning of the phrase "raising to a power" is extended to include this set of operations, and even the function theory involved in the use of imaginary exponents. There can be, of course, no objection on the grounds of logic to such an extension of meaning through appropriate definition, but there remains an objection on the basis of the fitness of things: the same phrase is used to denote both the series of operations and a single component of this series. Thus "raising four to the minus three-halves power" includes as a necessary operation that of "raising four [or $\frac{1}{4}$ or $\pm 2$ or $\pm\frac{1}{2}$, depending on the order of operations indicated in the definition] to the third power." Here the phrase "raising to a power" is used in two entirely different senses. Moreover, the one sense is not a generalization of the other. One might equally well speak of $4^{-3/2}$ as "finding a minus-three-halves

[1] These are substantially the definitions given also in the *Mathematics Dictionary* (Ed. by Glenn James, Van Nuys, Cal., 1942), the *Mathematisches Wörterbuch*, (by G. S. Klügel, 5 vols., Leipzig, 1803-1831), and the *Encyclopedie des Sciences Mathématiques*, (Ed. by Jules Molk and Franz Meyer, vol. I (1), Paris, 1904, pp. 53-56.

[2] Cf., for example, W. L. Hart, *Plane Trigonometry* (New York, 1933), p. 21; J. B. Rosenbach, E. A. Whitman, David Moskovitz, *Plane Trigonometry* (New York, 1937), p. 139.

root" of four, or even as "taking a minus-three-halves reciprocal of four." The simplest procedure would appear to be to retain Webster's meaning of the word "power" and to substitute some other expression for the extended sense. The word "exponent" might well be reserved for this use, but there is a slight practical objection here also, as a short excursion into the past will show.

The close association in thought of the concept of power and the notation of exponents has led inadvertently to some confusion in the history of mathematics. Standard works on the subject[3] state that Nicole Oresme in the fourteenth century first used fractional exponents. In substantiation of this assertion they indicate that in the *Algorismus proportionum* one finds such expressions as

$$\frac{p1}{1 \cdot 2}$$

to denote what Oresme expressed as the three-halves "proportion" (i. e., the cube of the principal square root), so that $(\sqrt{4})^3$ might appear[4]

as
$$\frac{1p}{1 \cdot 2 \cdot 4} \text{ or } \frac{3p}{2 \cdot 4}.$$

Here are clear-cut examples of a not inconvenient notation for fractional "powers" as described above; but they do not illustrate the use of *exponents* in the ordinary sense as given by Webster. Oresme gave rules equivalent to such expressions as

$$(a^m)^{p/q} = (a^{mp})^{1/q} \text{ or } a^m \cdot a^{1/n} = a^{m+(1/n)};$$

but the statements of these are largely verbal rather than symbolic, and in no case does the indicator of the power or root appear in the position of a modern exponent.[5] The *idea* of fractional "powers"

[3] See, for example, Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. II (Leipzig, 1892), p. 121ff; Kark Fink, *A Brief History of Mathematics* (transl. by W. W. Beman and D. E. Smith, Chicago, 1910), p. 102; D. E. Smith, *History of Mathematics*, vol. I (Boston, 1923), p. 239; Johannes Tropfke, *Geschichte der ElementarMathematik*, vol. I (Leipzig, 1902), p. 200. Cf. also Maximilian Curtze, *Der algorismus proportionum des Nicolaus Oresme* (Berlin, 1868), p. 9ff; H. G. Funkhouser, "Historical development of tne graphical representation of statistical data" [*Osiris*, III (1937), 269-404), p. 274; Hermann Hankel, *Zur Geschichte der Mathematik im Alterthum und Mittelalter* (Leipzig, 1874), p. 350; *Encyclopédie des sciences mathématiques*, I(1), 56; Florian Cajori, *A History of Mathematical Notations*, (2 vols., Chicago, 1928-1929), I, 343, 354.

[4] Heinrich Wieleitner, "Zur Geschichte der gebrochenen Exponenten", *Isis*, VI (1924), 509-520. Cf. references to Cantor, Curtze, Fink, Hankel, Smith, and Tropfke above; also Florian Cajori, *A History of Mathematics* (2nd ed., New York, 1931), p. 127.

[5] See Wieleitner, *op. cit.*, Curtze, *op. cit.*

or "proportions" quite possibly goes back long before the time of Oresme, but the *notation* of fractional exponents did not appear until several hundred years later. It is not unlikely that the Scholastic doctrine of fractional proportions may some time be traced through Arabic treatises and Greek works on arithmetic back to Pythagorean musical theory.[6] At any rate, Oresme was not the earliest medieval scholar to deal with fractional "powers", for Thomas Bradwardine in his *Liber de proportionibus* of 1328 had referred to "medietas duplæ proportionis" and "medietas sesquioctavæ proportionis"[7] (i. e., $\sqrt{2}$ and $\sqrt{9/8}$). Nevertheless it may be that Oresme first discussed proportions made up of both powers and roots. Moreover, he appears to have been the first one to represent such proportions symbolically. There is at hand a convenient word, *index*, which might well be used to denote all such symbolisms, for Webster defines it as "The figure, letter, or expression showing the power or root of a quantity." This would correctly characterize the notation of Oresme without giving the false impression that here one finds *exponents* in the strict sense.

The use of exponents as indicators of positive integral powers is widely ascribed to Descartes, but in reality this goes back a century and a half before his time. Nicolas Chuquet in 1484 composed a *Triparty en la science des nombres* which was probably inspired by the work of Oresme of about a century before. In the *Triparty* there are expressions such as $.5.^1$ and $.6.^2$ and $.10.^3$ to designate what now would appear as $5x$ and $6x^2$ and $10x^3$. Here powers are clearly indicated by exponents, although Chuquet used the word *denominacion* instead of *potence* and his form differs slightly from the modern Cartesian notation.[8] In this remarkable work negative integers and zero also are used as exponents: 9 (or $9x^0$) is written as $.9.^0$ and one reads correctly that $.72.^1$ divided by $.8.^3$ is $.9.^{2\cdot m}$ (i. e., $72x \div 8x^3 = 9x^{-2}$). Chuquet possessed also a brief notation for roots,—such as $R^2$ .7. for the square root of 7 and $R^4$ .10. for the fourth root of 10,— but this corresponded to our form $\sqrt[2]{7}$ and $\sqrt[4]{10}$ rather than to the fractional-exponent type in $7^{1/2}$ and $10^{1/4}$.

Tradition has attributed the earliest use of fractional exponents to John Wallis, but this should be interpreted cautiously. In 1585 in the *Arithmétique* of Simon Stevin powers of one-tenth and powers of

[6] See Heinrich Wieleitner, "Geschichte der gebrochenen Exponenten", *Isis*, VII (1925), 490-491.

[7] See Wieleitner, *op. cit.* (1924), p. 515.

[8] For Chuquet's work see Aristide Marre, "Notice sur Nicolas Chuquet et son Triparty en la science des nombres," *Bulletino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, XIII (1880), 555-659, 693-814, especially pp. 737ff; and Ch. Lambo, "Une algèbre française de 1484. Nicolas Chuquet," *Revue des Questions Scientifiques* (3), II (1902), 442-472, especially pp. 459-463.

unknowns were denoted by figures, frequently encircled, placed either over or after the digit or the coefficient. Thus the number 6.789 might appear as $\overset{0}{6}\,\overset{1}{7}\,\overset{2}{8}\,\overset{3}{9}$, and the polynomial $1+2x+3x^2+4x^3$ could be expressed by 1 ⓪+2 ①+3 ②+4 ③. Stevin indicated clearly that this notation could be extended readily to include all roots:

> Toutefois le 1/2 en circle seroit le charactere de racine de ①, & par consequent 2/3 en un circle seroit le charactere de racine quarrée de ③, par ce que telle 3/2 en circle multipleé en foi donne produict ③, & ainsi des austres; de sorte que par tel moyen on pourroit de toutes simples quantitez extraire especes de racines quelconques, comme racine cubique de ② seroit 2/3 en circle, etc.[9]

Such notations are clearly equivalent to fractional exponents, but literal-minded readers will notice that the indices were placed over *or* on the right, rather than above *and* on the right, as common usage and the definition of Webster require. For fractional exponents in the strict sense one waits almost another century.

The analytic geometry of Descartes in 1637 popularized the use of positive integral exponents in the modern manner.[10] This work exerted a strong influence upon John Wallis, who applied the ideas and notations in his *Arithmetica infinitorum* of 1655. Here Wallis proposed his well-known principle of interpolation or of (incomplete) induction, asserting that inasmuch as the area under the curves $y=x^n$ was given by the expression

$$\frac{x^{n+1}}{n+1}$$

for all integral values of $n$, this formula was seen, by analogy, to hold also for fractional and even irrational values of the exponent![11] The period of mathematical rigor was yet a century and a half removed. Throughout the treatise he uses integral exponents in the strict modern sense and speaks freely of fractional and irrational indices. These latter apparently first actually appeared, however, in Newton's famous

[9] Simon Stevin, *Les Oeuvres mathématiques* (ed. by Albert Girard, Leiden 1634), I, 6. See also Eugène Prouhet, "Sur l'invention des exponants fractionnaires ou incommensurables," *Nouvelles Annales de Mathématiques*, vol. XVIII (1859), *Bulletin de Bibliographie*, pp. 42-46. Cf. also Tropfke, *op. cit.*, p. 102. Joost Bürgi made use of a similar form, writing indices of positive integral powers of the unknown as Roman numerals placed over the corresponding coefficients. It should perhaps be remarked also that Stevin's notation for polynomials resembles somewhat that used earlier by Bombelli and later by Girard. See Cajori, *A History of Mathematical Notations*, I, 343-360.

[10] Roman numerals had been used as exponents the year before by James Hume who wrote $A^3$ as $A^{iii}$. See Cajori, *A History of Mathematical Notations*, I, 345f.

[11] See John Wallis, *Opera mathematica* (2 vols., Oxonii, 1656-1657), II, 52-53.

letter to Oldenburg of June 13, 1676.[12] In Wallis' *Algebra* of 1685 fractional exponents, both positive and negative, appear frequently.[13]

The use of fractional exponents quickly became common practice. Newton stated the binomial theorem so as to include all (rational) exponents, and numerous other formal expressions involving positive integral powers were found to be satisfied also when these integers were replaced by corresponding negative or fractional or even irrational values. Leibniz, in a letter to Wallis, suggested the possibility of fractional derivatives and integrals. The operation of the principle of the permanence of form tended to obscure the fact that while the type of *expression* remained essentially the same, the entire *meaning* had been radically altered. The exponent in $x^n$ indicates a continued multiplication if $n$ is a positive integer, but not otherwise. In the days of Bradwardine and Oresme the very same word had been used for powers and roots: *proportio dupla* meant varying as the square, *proportio subdupla* signified varying as the square root. Hence the term *proportio* was naturally carried over to all fractional indices, and the index $1\frac{1}{2}$ denoted *proportio sesquialtera*. As the Greek and Latin emphasis upon the idea of proportion gradually gave way to the development in terms of the arithmetic operations as now defined, powers and roots were more clearly dissociated. Whereas Oresme regarded the index 2/3 as designating *one proportion*, Stevin explicitly stated that it denoted *two distinct operations*, involution and evolution. However, the success of logarithms, Wallis' principle of induction, and the excessive formalism in the calculus of Leibniz, tended to obscure this distinction. No essential difference was seen in the expression $x^3$ and $x^{\sqrt{3}}$, for algorithmic rules applied to them indifferently. This tendency remains to the present day in ever so many textbooks which prove the laws of exponents for positive integral powers and then, with no warning or apology to the reader, treat these laws as adequately justified for entirely different situations indicated by real indices, integral or fractional, rational or irrational. Such specious procedures are encouraged by the careless use of the phrase, "raising to a power". Having raised four to the second power, a beginner experiences a comfortable feeling of understanding when he is told that similarly ten "raised to the .30103...th power" is equal to two. A clear-cut distinction between the terms index, exponent, and power would go far toward ameliorating the bliss of such ignorance. Incidentally it would serve also to render less obscure certain historical situations.

[12] See Isaac Newton, *Opera quæ exstant omnia* (ed. by Samuel Horsley, Londini, 1779-1785), IV, 215.

[13] See John Wallis, *A Treatise on Algebra* (London, 1685), p. 332; cf. pp. 310, 319, and *passim*.

One then would be able to state unequivocally, so far as extant evidence permits, that the idea of fractional "proportions" was referred to by Bradwardine and probably was of much earlier origin; that these appear first to have been represented symbolically by indices by Oresme; that exponents were adopted for integral indices by Chuquet and popularized in modern form by Descartes; that fractional exponents were adumbrated by Stevin and effectively established by Wallis and Newton; and that fractional "powers" are no longer *comme il faut.*

# *Problem Department*

*Edited by*
E. P. Starke and N. A. Court

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to Emory P. Starke, Rutgers University, New Brunswick, N. J.

## *SOLUTIONS*

No. 494. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find the smallest possible base of a system of numeration in which the three-digit number 777 is a perfect fourth power.

Solution by *J. Ernest Wilkins, Jr.*, Tuskegee Institute.

Let $a$ be the base; then $7a^2+7a+7=b^2$ for some integer $b$. It follows that $b$ is a multiple of 7, $b^4$ is a multiple of $7^4$, and $a^2+a+1\equiv 0$ (mod 343), or upon multiplying by 4,

(1) $$(2a+1)^2\equiv -3\equiv 340 \pmod{343}.$$

To solve this congruence, put $2a+1=c+7d+49e$, where $c$, $d$, $e$ are non-negative integers $\leqq 6$. Then we have

(2) $$c^2+14cd+49(d^2+2ce)\equiv 340 \pmod{343}.$$

Consequently, $c^2\equiv 4$ (mod 7), so that $c$ is 2 or 5. Taking $c=2$, we find that the congruence (2) becomes

(3) $$4d+7d^2+28e\equiv 48 \pmod{49}.$$

Consequently $4d\equiv 48$ (mod 7), so that $d=5$. Then relation (3) reduces to $e\equiv 0$ (mod 7) so that $e=0$. Thus $a=18$. Since the two incongruent roots of (1) have a sum congruent modulo 343 to $-1$, the next smallest positive root of (1) is $a=324$. Therefore 18 is the smallest possible solution of the necessary condition (1). Actually, in base 18, $777=7^4$. Thus 18 is the desired solution.

Also solved by *Immanuel Marx* and *J. Szmojsz*.

No. 495. Proposed by *J. P. Wagman*, Washington, D. C.

Let $A=(3\sqrt{3}+5)^{2n+1}$, and let $F$ be the fractional part of $A$. Show that $2AF=4^{n+1}$.

Solution by *J Frank Arena*, Hardin, Illinois.

Let $I$ be the integral part of $A$, so that

$$A=I+F=(3\sqrt{3}+5)^{2n+1}.$$

Now
$$F'=(3\sqrt{3}-5)^{2n+1}$$

is positive and less than 1. We have at once

$$I+F-F'=(3\sqrt{3}+5)^{2n+1}-(3\sqrt{3}-5)^{2n+1}$$
$$=2({}_{2n+1}C_1(3\sqrt{3})^{2n}\cdot 5+{}_{2n+1}C_3(3\sqrt{3})^{2n-2}\cdot 5^3+\cdots).$$

Since this result is an integer, it evidently follows that $F'=F$. Thus finally

$$2AF=2[(3\sqrt{3}+5)(3\sqrt{3}-5)]^{2n+1}=2(27-25)^{2n+1}=2^{2n+2}=4^{n+1}.$$

No. 497. Proposed by *E. P. Starke*, Rutgers University.

Given any real number $N_0$. If $N_{j+1}=\cos N_j$, show that the limit of $N_j$ as $j\to\infty$ is a fixed number independent of $N_0$, and find an approximation of its value.

Solution by *M. S. Robertson*, Rutgers University.

Let $N=0.7391\ldots$ be the real root of $N-\cos N=0$, computed by Newton's method or otherwise. First note the following evident relations:

(1) $$-1\leqq N_1\leqq 1,\quad 0<N_j\leqq 1 \text{ for } j>1;$$

(2) $$-\tfrac{1}{2}<\tfrac{1}{2}(N_1+N)<1,\quad 0<\tfrac{1}{2}(N_j+N)<1 \text{ for } j>1;$$

(3) $$|\sin\tfrac{1}{2}(N_j-N)|\leqq\tfrac{1}{2}|N_j-N|;\quad |\sin\tfrac{1}{2}(N_j+N)|<\sin 1 \text{ for } j>0.$$

By elementary trigonometry and (3) it is easy to establish

(4) $$|N_{j+1}-N|=|\cos N_j-\cos N|=|2\sin\tfrac{1}{2}(N_j+N)\cdot\sin\tfrac{1}{2}(N_j-N)|.$$

Thus

(5) $$|N_1-N|\leqq|N_0-N|,\quad |N_{j+1}-N|\leqq(\sin 1)\cdot|N_j-N| \text{ for } j>0.$$

Now an easy induction on $j$ gives

$$|N_j-N|\leqq(\sin 1)^{j-1}|N_1-N|\leqq(\sin 1)^{j-1}|N_0-N|,$$

whence the limit of $N_j$ is $N=0.7391$.

Also solved by *J. Ernest Wilkins, Jr.*

No. 498. Proposed by *F. C. Gentry*, Louisiana Polytechnic Institute.

Prove
$$\begin{vmatrix} \cos B \cos C & \cos A \cos C & \cos A \cos B \\ \cos A & \cos B & \cos C \\ bc & ac & ab \end{vmatrix} = 0,$$

where, as usual, the letters represent angles and sides of any triangle.

Solution by *J. Szmojsz*, New York City.

Upon substitution of $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$ in the third row, the determinant becomes

$$\Delta = 4R^2 \begin{vmatrix} \cos B \cos C & \cos A \cos C & \cos A \cos B \\ \cos A & \cos B & \cos C \\ \sin B \sin C & \sin A \sin C & \sin A \sin B \end{vmatrix}.$$

If the elements of the first row be subtracted from the corresponding elements of the third row, the third row becomes

$$-\cos(B+C) \quad -\cos(A+C) \quad -\cos(A+B).$$

As $A+B+C = 180°$, the last two rows are now identical, whence $\Delta = 0$.

Also solved by *J. F. Arena, Lucio Chiquito, C., D. L. MacKay, P. D. Thomas, Leon Shenfil, W. Roy Utz*, and the *Proposer*.

No. 499. Proposed by *F. C. Gentry*, Louisiana Polytechnic Institute.

If $A$, $B$, $C$ are the angles of a triangle show that

$$2 + 2 \cos A \cos B \cos C = \sin^2 A + \sin^2 B + \sin^2 C$$

and hence show that the determinant

$$\begin{vmatrix} 0 & y + x \cos C & z + x \cos B \\ x + y \cos C & 0 & z + y \cos A \\ x + z \cos B & y + z \cos A & 0 \end{vmatrix}$$

$$= (x \sin A + y \sin B + z \sin C)(yz \sin A + zy \sin B + xy \sin C),$$

identically in $x$, $y$ and $z$.

Solution by *J. Frank Arena*, Hardin, Illinois.

The preliminary statement may be shown as follows:

$$\sin^2 A + \sin^2 B + \sin^2 C = \tfrac{1}{2}(1 - \cos 2A + 1 - \cos 2B) + 1 - \cos^2 C$$
$$= 2 - \tfrac{1}{2}(\cos 2A + \cos 2B) - \cos^2(A+B)$$

$$=2-\cos(A+B)\cos(A-B)-\cos^2(A+B)$$
$$=2-\cos(A+B)[\cos(A-B)+\cos(A+B)]$$
$$=2-\cos(A+B)\cdot 2\cos A\cos B.$$

which is the proposed relation since $\cos C=-\cos(A+B)$.

The expansion of the determinant is

$$xyz(2+2\cos A\cos B\cos C)+x(y^2+z^2)(\cos B\cos C+\cos A)$$
$$+y(z^2+x^2)(\cos C\cos A+\cos B)+z(x^2+y^2)(\cos A\cos B+\cos C).$$

The product of the two trinomials is

$$xyz(\sin^2 A+\sin^2 B+\sin^2 C)+x(y^2+z^2)\sin B\sin C$$
$$+y(z^2+x^2)\sin C\sin A+z(x^2+y^2)\sin A\sin B.$$

The first pair of coefficients are equal by the proof given above. Since

$$\cos B\cos C+\cos A=\cos B\cos C-\cos(B+C)=\sin B\sin C,$$

the second pair are equal. Similarly the other pairs are equal and the statement is an identity.

Also solved by *D. L. MacKay*, *Leon Shenfil*, *J. Szmojsz*, *P. D. Thomas*, and the *Proposer*.

No. 503. Proposed by *William E. Taylor*, Student, Colgate University.

$A$ and $B$, with abscissas $a$ and $b$ respectively, $(0<a<b)$, are points of the curve $x^2y=1$. Consider the rectangle having sides parallel to the coordinate axes and with $A$ and $B$ for a pair of opposite vertices. Prove that the curve divides the area of this rectangle in the ratio $b:a$.

Solution by *James Stewart*, Student, John McNeese Junior College, Lake Charles, Louisiana.

Th etwo areas, $K_1$ and $K_2$, are easily found by integration. They are:

$$K_1=\int_a^b (1/a^2-1/x^2)dx=(b-a)^2/a^2b,$$

$$K_2=\int_a^b (1/x^2-1/b^2)dx=(b-a)^2/ab^2.$$

The proposition then follows immediately.

Also solved by *Earl Greer*, *Leon Shenfil*, *Paul D. Thomas*, *W. Roy Utz*, and the *Proposer*.

No. 504. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle such that an altitude, a median, and an external bisector of an angle shall be concurrent.

Solution by *Paul D. Thomas*, United States Coast and Geodetic Survey.

Choose one side, say $AC$, and an adjacent angle $ACR$. Draw the external bisector of $ACR$. From $A$ drop a perpendicular upon $CR$ meeting the bisector in $H$. From $S$, the midpoint of $AC$, draw $HS$ meeting $CR$ in $B$. $ABC$ is the required triangle.

Also solved by *D. L. MacKay*, and the *Proposer*.

Another solution may be found as No. 354 in the December, 1940, issue of this Magazine.

No. 507. Proposed by *V. Thébault*, Tennie, Sarthe, France.

In a tetrahedron the sum of the squares of the altitudes is not greater than four ninths of the sum of the squares of the edges.

Solution by *P. W. Allen Raine*, Newport News (Va.) High School.

Let $h_i$ be the altitudes and $m_i$ the medians ($i=A, B, C, D$) from the corresponding vertices of the tetrahedron $ABCD$. Then it is obvious that

$$h_A^2+h_B^2+h_C^2+h_D^2 \leqq m_A^2+m_B^2+m_C^2+m_D^2.$$

Now, the square of any median equals 1/3 the sum of the squares of the three adjacent edges less 1/9 the sum of the squares of the three remote edges. (See Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 57. The Macmillan Co. 1935). This gives

$$m_A^2+m_B^2+m_C^2+m_D^2=4(AB^2+AC^2+AD^2+BC^2+BD^2+CD^2)/9$$

which proves the proposition.

Also solved by *D. L. MacKay*, *Paul D. Thomas*, and the *Proposer*.

## *PROPOSALS*

No. 535. Proposed by *Pvt. L. Lawrence*, A. S. T. P., Rutgers University.

A marble starts rolling at the top of a sphere under the influence of gravity. Where will it leave the surface of the sphere? (Neglect friction and air resistance.)

No. 536. Proposed by *E. Hoff*.

Within a time $T$, a train must pass a certain signal tower at a distance $D$. Its initial velocity is zero. Suppose the upper bound of acceleration is $M''$ and is fixed. Let $M'$ be the upper bound of velocity. There is a lower bound to the values of $M'$ which is greater than the average velocity $D/T$. Compute this lower bound for $M'$.

No. 537. Proposed by *E. P. Starke*, Rutgers University.

There are just two values of $n(n>10)$ for which the sum of the squares $10^2+11^2+\cdots+n^2$ is exactly divisible by the sum of the numbers $10+11+\cdots+n$. Find them.

No. 538. Proposed by *H. T. R. Aude*, Colgate University.

While working on a triangle problem a student noticed that the three given sides were represented by three relatively prime integers in arithmetic progression. He solved the problem and found that one of the angles was twice as large as another. Find the triangle. 4, 5, 6.

No. 539. Proposed by *Paul D. Thomas*, United States Coast and Geodetic Survey.

$P$ and $Q$ are the extremities of a pair of conjugate diameters of an ellipse, center $O$. Find (a) the locus of the centroid of the triangle $OPQ$, and (b) the locus of the foot of the perpendicular from $O$ upon $PQ$.

No. 540. Proposed by *N. A. Court*, University of Oklahoma.

The planes $(M_a)$, $(M_b)$, $(M_c)$ passing through the Monge point of a tetrahedron $(T)$ and perpendicular to the bimedians $m_a$, $m_b$, $m_c$ cut the bimedians $m_b$ and $m_c$, $m_c$ and $m_a$, $m_a$ and $m_b$ in the points $P_b$ and $P_c$, $Q_b$ and $Q_a$, $R_a$ and $R_b$. If $G$ is the centroid of $(T)$ and $k^2$ the sum of the squares of the edges of $(T)$, we have

$$GQ_a\cdot GR_a+GR_b\cdot GP_b+GQ_c\cdot GP_c=k^2/16.$$

No. 541. Proposed by *N. A. Court*, University of Oklahoma.

In the face $ABC$ of a tetrahedron $DABC$, find a point $M$ such that area $MBC$ : area $MCA$ : $MAB$ = area $DBC$ : area $DCA$ : area $DAB$.

# *Bibliography and Reviews*

*Edited by*
H. A. SIMMONS and P. K. SMITH

*Plane and Spherical Trigonometry. Alternate Edition.* By Lyman M. Kells, Willis F. Kern, and James R. Bland. McGraw-Hill Book Company, New York, 1943. xvii+400 pages.

This alternate edition is another manifestation of the strenuous effort on the part of the Naval Academy to fill the need today for mathematics texts adapted to war conditions.

The first three chapters cover the usual material through the solution of right triangles with logarithms. In this material is included the solution of the right triangle by the slide rule. In the applications of the right triangle to rectilinear figures in Chapter IV the subject of piloting is taken up.

Chapters V through X, inclusive, treat the usual material included in the latter part of a standard course in plane trigonometry. Chapter XI is devoted to complex numbers along with a treatment of hyperbolic functions. Chapter XII gives a treatment of logarithms. The slide-rule solutions of the general triangle is the subject of the last chapter of the plane trigonometry.

The right spherical triangle and its applications are given in Chapters XIV and XV. In these applications parallel, plane, and middle-latitude sailing, and the principles of the mercator chart are treated.

The oblique spherical triangle is solved by use of Napier's rules in Chapter XVI. The oblique spherical triangle is solved by the use of the half-angle formulas and Napier's analogies in Chapter XVII.

The text ends with Chapter XVIII which covers nautical astronomy and introductory celestial navigation. The celestial sphere, time, latitude by the meridian altitude, and the fundamental process of navigation by use of the lines of position are treated.

A mastery of this text would afford a splendid basis for a course in navigation. The format of the text is attractive and the figures should motivate interest.

*Louisiana Polytechnic Institute.* P. K. SMITH.

*Intermediate Algebra for College Students.* By Thurman S. Peterson. Harper and Brothers Publishers. New York, 1942. viii+358 pages.

The author states in his preface that this book is designed to serve as a text for college students who have had not more than one year of secondary school algebra. Writing for this type of students, he has very clearly explained the underlying principles of all operations and rules except in a very few places.

In Chapter I (Introduction), the author, in a very clear manner, leads the reader, who knows operations with arithmetic numbers, to understand elementary algebraic operations with general numbers. Definitions are also clearly explained.

Chapters II and III are devoted to elementary operations with signed numbers. The use of negative numbers is presented very effectively.

"Equations and Stated Problems" and "Factoring" are discussed in Chapters IV and V. The presentation of several types of stated problems is quite thorough. The equations discussed are of first degree, involving only one unknown. In the discussion of fractions in Chapter VI he does not prove all the rules; he merely states them, suggesting that these rules are the same as corresponding rules in arithmetic.

In Chapter VII equations of the first degree, involving two and three unknowns, including those in terms of reciprocals of the unknown, are discussed. A good selection of stated problems, leading to equations of the first degree, is given.

"Exponents, Roots and Radicals", also equations involving radicals, are presented in Chapter VIII; then a short discussion of "Graphical Methods" is given in Chapter IX before taking up "Quadratic Equations" in Chapter X. "Systems Involving Quadratics" are presented in Chapter XI where the graphical method of solution is thoroughly explained.

The last three chapters "Ratio, Variation, Binomial Theorem", "Logarithms", and "Progressions" are presented in the usual manner, except that the discussion of proportion on page 275 is rather limited. On page 283, example 3, the suggestion is made that both sides of the equation $b^{3/2}=4$ be raised to the 2/3 power. This does not agree with his definition of a power on page 2, § 8.

The text is carefully written. The method of giving prominence to rules and illustrative problems is very effective. Stated problems are numerous and selections good. A review of each chapter is placed at its close. Answers to odd numbered problems are given, and the book closes with an index. The text is recommended for examination to teachers of intermediate algebra.

*Trinity University.* GEORGE A. NEWTON.

*Plane Trigonometry, Solid Geometry, and Spherical Trigonometry.* By Walter W. Hart and William L. Hart. D. C. Heath and Co., Boston, 1942. vii+280 pages. With logarithmic and trigonometric tables by William L. Hart. 124 pages.

The text was designed to meet the needs of people who are studying mathematics in order to apply the material to various aspects of the war effort. The inclusion of a section on solid geometry in a text on plane and spherical trigonometry is unusual, but the idea is sound and has been handled with skill by the authors. The reviewer hopes that the integration of these three branches of mathematics will not be allowed to lapse with the conclusion of the present emergency. The sections on plane and spherical trigonometry were written by William L. Hart and the section on solid geometry was prepared by Walter W. Hart.

The section on plane trigonometry deals with the acute angle, logarithms, the right triangle, and applications of the right triangle before discussing the general angle. This was indeed a wise choice, for it causes the student almost at once to feel the power and wide application of the subject and his interest is tremendously aroused. In the chapter on applications, there is not only a wealth of interesting problems, but the author has gone to great pains to clearly explain the physical principles involved, to define the terms introduced from other fields, and to set up and work out illustrative examples for each application.

Following applications of the right triangle, appear chapters on the functions of a general angle, radian and mil measure, variation and graphs of the functions, simple identities and equations, addition formulas, oblique triangles and applications, some advanced topics on identities and equations, and finally polar co-ordinates. The chapter on radian and mil measure is disturbing. It seems misplaced. It seems to have been fitted into the text where it was expected to do the least harm—and then radians

and mils were forgotten and never referred to again. Now students who are planning to work in field artillery, coast artillery, anti-aircraft fire, and other branches of the armed forces must learn to think not only in terms of degrees but in terms of mils as well. From my own experience, I know that neither I nor my students acquired this habit by just working a few problems in which the angles were expressed in mils. The same is true for those who must learn to think of angles measured in radians. It should not be difficult to overcome this mishap. The chapter on measurement of angles could be introduced quite early in the text and many of the applications, which are problems taken from artillery fire and gunnery, could be restated in terms of mils. It would also be necessary to add a complete table of the natural trigonometric functions and the logarithms of the trigonometric functions with the angles expressed in mils. There is also a serious error in this chapter. On page 81, the angular velocity, $\omega$, is correctly defined, but further down the page, one reads, "If $\omega$ is measured in radians", and on page 82, again one reads "$\omega=5\times 2\pi=10\pi$ radians". As a group, I think we mathematicians are often careless in stating units correctly and precisely. This is especially true when we are speaking to each other informally. Naturally we carry over these errors into our writing. Since this text is for students who want to apply mathematics to other fields, I hope these errors in units will be corrected at the earliest opportunity.

In the chapter on graphs, the author consistently uses units of measurement equivalent to 90° in his figures. The student is consistently told to draw graphs of the trigonometric functions between limits measured in degrees. But what will the student do when he reaches the problem $y=x+\sin x$ on page 96? The hint given in the footnote will not help him in his dilemma. More emphasis on units and radian measure would have saved the student considerable trouble.

The section on solid geometry consists of three chapters, entitled, "Planes and Polyhedral Angles", "Spherical Geometry", and "Measurement of Solids". The aim is to give the student a conception of surfaces and solids in space and to acquaint him with some of the most useful theorems of three dimensional geometry. The aim is realized by having the student complete proofs of theorems and by including long lists of good problems at the ends of chapters. A very interesting discussion of three commonly used maps of the earth is included in the chapter on spherical geometry. It is in this section of the book that I found great difficulty with expression. Perhaps I am unduly critical, but I must confess that I do not like to use the words "perpendicular" and "tangent" as nouns when I am discussing the geometry of three dimensional space. I prefer tangent line or tangent plane, perpendicular line or perpendicular plane. On page 157, section 134, I should prefer to read, "The distance from a point to a plane is the length of the line drawn perpendicular to the plane from the point". On page 171, the statement "All radii and all diameters of the same sphere are equal" is ambiguous. A number of people, mathematicians and non-mathematicians, to whom I have read that statement agree that one could conclude that the radius is equal to the diameter. On page 175, problem 6, the letter $N$ is used in two different senses. On page 176, I learn that it is permissible to say that an angle equals an arc when the angle is measured in angular degrees and the arc is measured in arc degrees. But unfortunately I have never seen the definition of an arc degree. Nor can I find its definition in this text. Would not the chapter on radian measure be useful here? On page 179, could the four points of the spherical polygon $ABCD$ be joined in any other order, say $ACBDA$? On page 193, I think the student would understand the usual meaning of the word conformal as applied to maps, namely, angles are preserved.

The section on spherical trigonometry consists of chapters on right spherical triangles, oblique spherical triangles, and applications. There is a clear discussion of navigation, including plane sailing, parallel sailing, middle latitude sailing, dead reckon-

ing, and great circle sailing. The celestial sphere and the astronomical triangle are adequately discussed and the section closes with some general remarks on celestial navigation. On page 239, in figure 13, the distance $AB$ should be designated "$d$" instead of "$a$". On pages 241 and 242, there is a tendency to use the term miles instead of the correct unit, nautical miles.

Throughout the text the authors have taken a sensible attitude toward the problem of errors and significant figures. However it is amusing to note that while $\pi = 22/7$ in solid geometry, $\pi = 3.1416$ in trigonometry. On the other hand, one nautical mile equals 6080.27 feet in solid geometry, whereas one nautical mile equals 6080 feet in spherical trigonometry.

In general, the text is sound and has been well written. No doubt it will be widely used. Some of my criticisms have reflected my own personal prejudices, but many have been occasioned by the haste in which the book was undoubtedly written. I sincerely hope that the authors will find the leisure in their busy lives to revise this text and bring it completely up to the excellent standards they have set in their previous writings.

*University of Illinois.* J. WILLIAM PETERS.

*Principles of College Algebra.* By Morris S. Knebelman and Tracy Y. Thomas. Prentice-Hall, Inc., New York, 1942. x+380 pages.

The authors of this book state in the preface that they will accent fundamental definitions and theorems with the intention, in part, of preventing rote study. Story problems and historical references are left out.

The chapter headings give some idea of the scope: (I) Real numbers and their properties, (II) Linear equations, (III) Polynomials, (IV) Partial fractions, (V) Exponents and radicals, (VI) Logarithms, (VII) Complex numbers, (VIII) Quadratic equations, (IX) Determinants and matrices, (X) The theory of equations, (XI) The cubic and quartic equations, (XII) Permutations and combinations, (XIII) Probability, (XIV) Mathematical induction, the binomial theorem, (XV) Sequences, limits and series. Each chapter ends with a set of problems, many of which were selected from recent examinations in American colleges and universities. At the end of the text, ten typical examinations are given.

Chapters I, II, III, IV, IX are least like what we expect to find in the usual college algebra of the past twenty years. The arithmetical problems of the division transformation, highest common factor, least common multiple, and prime numbers form a familiar background for the corresponding problems in the theory of polynomials. The theory of partial fractions is backed up by proofs. The authors might have had in mind the dilemma of the inquiring calculus student referred to his college algebra for an explanation of partial fractions, only to find almost nothing there. Linear dependence appears in Chapters II, IX.

On page 16 *inverse* appears instead of *converse*. Article 13 has several misprints in the inequality signs. Example 4, page 26, should read $b-c$ instead of $a-c$.

This book should prove stimulating to students as well as to instructors. It would be interesting to see other texts written in the same spirit.

*Virginia Military Institute.* WILLIAM E. BYRNE.